Recap of 16.2 (briefly!)

Line integrals: suppose a curve C is parameterized by $r(t) = \langle x(t), y(t) \rangle$, $\underline{a \le t \le b}$

① $\int_C f(x,y)\,\underline{ds} = \int_a^b f(r(t))\,|r'(t)|\,dt$

set up $\int_{\boxed{C}} (x^3 + xy^2)\,ds$, where $\vec{r}(t) = \langle t^4, 3t^2 \rangle$, $0 \le t \le 2$

$\int_C (x^3 + xy^2)\,ds = \int_0^2 f(r(t))\,|r'(t)|\,dt$ $\quad \vec{r}'(t) = \langle 4t^3, 6t \rangle$

$|r'(t)| = \sqrt{16t^6 + 36t^2}$

$= \int_0^2 \underbrace{\left[(t^4)^3 + t^4 (3t^2)^2 \right]}_{f(r(t))} \underbrace{\sqrt{16t^6 + 36t^2}\,dt}_{|r'(t)\,dt| = ds}$

② Find $\int_C xy\,\underline{dx} + y^2 x\,\underline{dy}$ if $r(t) = \langle \sin t, \cos t \rangle$, $0 \le t \le \frac{\pi}{3}$

$x = \sin t$, $dx = \cos t\,dt$

$y = \cos t$, $dy = -\sin t\,dt$

$\int_0^{\pi/3} (\sin t)(\cos t)\cos t\,dt + (\cos^2 t)(\sin t)(-\sin t)\,dt$

$\int_0^{\pi/3} \left(\sin t \cos^2 t - \cos^2 t \sin^2 t \right) dt$

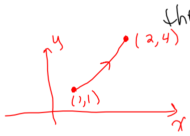
③ $\int_C F \cdot dr$ $\quad \vec{r}(t) = \langle x(t), y(t) \rangle$, $a \le t \le b$

$\int_a^b F(\vec{r}(t)) \cdot \vec{r}'(t)\,dt$ = WORK!

ex: Find the work done by the force field $F(x,y) = \langle x^4 y, y^3 \rangle$ in moving an object along the curve $\boxed{y = x^2}$ from the point $(1,1)$ to the point $(2,4)$

$W = \int_C F \cdot dr$

Parameterize C:



$\boxed{r(t) = \langle t, t^2 \rangle}$

$1 \le t \le 2$

$F(x,y) = \langle x^4 y, y^3 \rangle$

$\int_C F \cdot dr = \int_1^2 F(r(t)) \cdot r'(t)\,dt$

$= \int_1^2 \langle (t^4)(t^2), (t^2)^3 \rangle \cdot \langle 1, 2t \rangle\,dt$

$\int_1^2 \langle t^6, t^6 \rangle \cdot \langle 1, 2t \rangle\,dt$

$\int_1^2 (t^6 + 2t^7)\,dt$

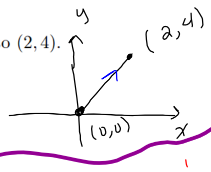
⋮

Section 16.3 Fundamental Theorem for Line Integrals

In section 16.2, we learned how to find a line integral over a vector field **F** along a curve C that is parametrized by $\mathbf{r}(t)$, $a \le t \le b$: $\int_c \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$.

Example 1: Suppose we are moving a particle from the point $(0,0)$ to the point $(2,4)$ in a force field $\mathbf{F}(x,y) = \langle y^2, x \rangle$. Find $\int_c \mathbf{F} \cdot d\mathbf{r}$ where:

a.) The particle travels along the line segment from $(0,0)$ to $(2,4)$.



step1: Parameterize C:

Line is parameterized by

$r(t) = \vec{r}_0 + t\vec{v}$

$r(t) = \langle 0,0 \rangle + t\langle 2,4 \rangle$

$r(t) = \langle 2t, 4t \rangle$, $0 \le t \le 1$

$r'(t) = \langle 2, 4 \rangle$

$F(x,y) = \langle y^2, x \rangle$

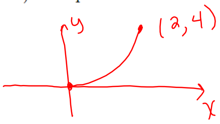
$\int_C F \cdot dr = \int_0^1 F(r(t)) \cdot r'(t)\,dt$

$= \int_0^1 \langle 16t^2, 2t \rangle \cdot \langle 2, 4 \rangle\,dt$

$= \int_0^1 (32t^2 + 8t)\,dt$

$= \boxed{\frac{44}{3}}$

b.) The particle travels along the curve $y = x^2$ from $(0,0)$ to $(2,4)$.



Parameterize $y = x^2$

$r(t) = \langle t, t^2 \rangle$, $0 \le t \le 2$

$r'(t) = \langle 1, 2t \rangle$

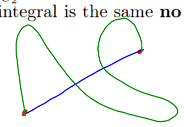
$F(x,y) = \langle y, x \rangle$

$\int_C F \cdot dr = \int_0^2 F(r(t)) \cdot r'(t)\,dt = \int_0^2 \langle t^4, t \rangle \cdot \langle 1, 2t \rangle\,dt$

$= \int_0^2 (t^4 + 2t^2)\,dt = \boxed{\frac{176}{15}}$

Note: Although the end points are the same, the value of the line integral is **different** because the **paths** are different. In this section, we will learn under what conditions the line integral is independent of the path taken.

Definition: If **F** is a continuous vector field, we say that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is **independent of path** if and only if $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for any two paths C_1 and C_2 with the same starting and ending points. In other words, the line integral is the same **no matter what path** you travel on as long as the endpoints are the same.



Definition: A vector field **F** is called a **conservative vector field** if it is the gradient of some scalar funtion f, that is there exists a function f so that $\mathbf{F} = \nabla f$. We call f the **potential function**.

Example 2. Consider $f(x,y) = x^2y - y^3$. Find the gradient, explain why it is conservative, and ^find the potential function.

scalar function

$\nabla f = \langle f_x, f_y \rangle = \langle 2xy, x^2 - 3y^2 \rangle$

why is $\langle 2xy, x^2 - 3y^2 \rangle$ a conservative vector field?

$\langle 2xy, x^2 - 3y^2 \rangle$ is conservative since $\nabla f = \langle 2xy, x^2 - 3y^2 \rangle$

and the potential function is $f(x,y) = x^2y - y^3$.

Recall the Fundamental Theorem of Calculus tells us that $\int_a^b f'(x)dx = f(b) - f(a)$. Since $\nabla f = \langle f_x, f_y \rangle$, we can think of the potential function, f, as some sort of antiderivative of ∇f. Hence $\int \mathbf{F}\cdot d\mathbf{r} = \int \nabla f \cdot d\mathbf{r}$.

Fundamental Theorem for Line Integrals: Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \le t \le b$. Let $\mathbf{F}$ be a conservative vector field. Let f be a differentiable function of two or three variables whose gradient vector, ∇f, is continuous on C. Then

$r(t) = \langle x(t), y(t) \rangle$

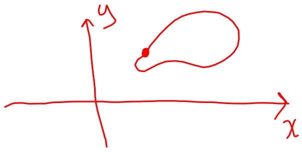
$a \le t \le b$

Recall: F is conservative if $F = \nabla f$, $f(x,y)$ is the potential function

$$\boxed{\int_c \mathbf{F}\cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))}$$

Note: **Line integrals of conservative vectors fields are independent of path** because in a conservative vector field, the line integral is computed by only using the **endpoints** of the domain! Therefore, if we are in a conservative vector field, the line integral along a curve C in that vector field will be the same **no matter what curve we travel across** that connects the endpoints together. **WHICH MEANS WE DON'T EVEN NEED TO PARAMETERIZE THE CURVE!**

Note: If $\mathbf{F}$ is a **conservative** vector field and C is a **closed path** (one in which the starting point and ending point is the same), then $\int_c \mathbf{F}\cdot d\mathbf{r} = 0$. Why?



closed path means the start point of the curve is the same as the end point. so, $r(b) = r(a)$

$$\int_c \vec{F}\,dr = \int_c \nabla f \cdot dr = f(r(b)) - f(r(a))$$

$$= 0$$

since $\vec{F}$ is conservative

Example 3: Let $f(x,y) = 3x + x^2y - yx^2$. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F} = \nabla f$ and C is the curve given by $\mathbf{r}(t) = \langle 2t, t^2 \rangle$, $1 \le t \le 2$.

$$\int_c F\cdot dr = \int_c \nabla f \cdot dr = f(r(2)) - f(r(1))$$

$r(2) = (4,4)$

$r(1) = (2,1)$

$$= f(4,4) - f(2,1)$$

since $f(x,y) = 3x + x^2y - yx^2$,

$f(4,4) = 12 + 64 - 64 = 12$

$f(2,1) = 6 + 4 - 4 = 6$

$= 12 - 6$

$= 6$

Question: How do we determine if a vector field is conservative, and if so, how do we find the potential function? The 'test for conservative' we use depends on whether **F** is in $\mathbb{R}^2$ or $\mathbb{R}^3$.

Theorem: $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle = P\mathbf{i} + Q\mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, if and only if $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$. Note: This above criteria to determine if a vector field is conservative works only for $\mathbb{R}^2$.

Short proof:

Suppose $F = \langle P, Q\rangle$ is conservative.

$F = \nabla f = \langle f_x, f_y\rangle$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ $\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$

Example 4: Is $\mathbf{F}(x,y) = \langle 3x^2 - 4y, 4y^2 - 2x\rangle$ a conservative vector field? If so, find a function f so that $\mathbf{F} = \nabla f$.

conservative test: does $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$?

$P = 3x^2 - 4y$

$Q = 4y^2 - 2x$

$\frac{\partial P}{\partial y} = -4$, $\frac{\partial Q}{\partial x} = -2$

F not conservative since $\frac{\partial Q}{\partial x} \neq \frac{\partial P}{\partial y}$!

so there is <u>no</u> function $f(x,y)$ whose gradient is F.

Example 5: Is $\mathbf{F}(x,y) = \langle \overset{P}{x+y}, \overset{Q}{x-2}\rangle$ a conservative vector field? If so, find a function f so that $\mathbf{F} = \nabla f$.

$P = x+y$ $\frac{\partial P}{\partial y} = 1$, $\frac{\partial Q}{\partial x} = 1$ yes! it's conservative.

$Q = x-2$

How do we find the potential function?!

$\vec{F} = \langle x+y, x-2\rangle$ is conservative means there is some function $f(x,y)$ such that

$\nabla f = \langle x+y, x-2\rangle$

$\langle f_x, f_y\rangle = \langle x+y, x-2\rangle$

this means $\int f_x = \int (x+y)\,dx$

and $\int f_y = \int (x-2)\,dy$

$\int (x+y)\,dx = \frac{x^2}{2} + yx + \boxed{g(y)}$

$\int (x-2)\,dy = xy - 2y + \boxed{h(x)}$

$f(x,y) = xy + \frac{x^2}{2} - 2y$

check it! $\nabla f = \langle f_x, f_y\rangle = \langle y+x, x-2\rangle = F$ ☺

Example 6: Given $\mathbf{F}(x,y) = \langle 2xy^3, 3x^2y^2\rangle$. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is the curve given by

$\mathbf{r}(t) = \langle t^3+2t^2-t, 3t^4-t^2\rangle$, $0 \le t \le 2$.

check to see if $F(x,y) = \langle \overset{P}{2xy^3}, \overset{Q}{3x^2y^2}\rangle$ is conservative.

$F = \langle P, Q\rangle$ is conservative if $\boxed{F = \nabla f}$

to test for conservativeness : Is $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$?

$\frac{\partial Q}{\partial x} = 6xy^2$, $\frac{\partial P}{\partial y} = 6xy^2$ yes!

To find $f(x,y)$, $\int 2xy^3\,dx = x^2y^3 + g(y)$

$\int 3x^2y^2\,dy = x^2y^3 + h(x)$

$$\boxed{f(x,y) = x^2y^3}$$

Example 6: Given $\mathbf{F}(x,y) = \langle 2xy^3, 3x^2y^2\rangle$. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is the curve given by

$\mathbf{r}(t) = \langle t^3+2t^2-t, 3t^4-t^2\rangle$, $0 \le t \le 2$.

$$\int_C F\cdot dr = \int_C \nabla f\, dr = f(r(2)) - f(r(0))$$

$r(2) = (8+8-2, 48-4) = (14, 44)$

$r(0) = (0,0)$

$= f(14,44) - f(0,0)$

$f(x,y) = x^2y^3$

$= (14)^2(44)^3$

Example 7: Let $\mathbf{F}(x,y) = \langle 3+2xy^2, 2x^2y\rangle$. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is the arc of the hyperbola $y = \frac{1}{x}$ from $(1,1)$ to $\left(4, \frac{1}{4}\right)$.

- F is not conservative, it is path dependent, meaning you must parameterize the curve!
- F is conservative, meaning it is path independent, only endpoints matter. (so we do not parameterize C)

Is $F(x,y) = \langle \underset{P}{3+2xy^2}, \underset{Q}{2x^2y}\rangle$ conservative?

$\frac{\partial Q}{\partial x} = 4xy$

$\frac{\partial P}{\partial y} = 4xy$

vector field is conservative. there is a function $f(x,y)$ so that $\langle 3+2xy^2, 2x^2y\rangle = \nabla f$

To find $f(x,y)$:

$\int (3+2xy^2)\,dx = 3x + x^2y^2 + g(y)$

$\int 2x^2y\,dy = x^2y^2 + \boxed{h(x)}$

$$\boxed{f(x,y) = x^2y^2 + 3x}$$

$f(4, \frac{1}{4}) = 16(\frac{1}{16}) + 12 = 13$

$f(1,1) = 4$

Example 7: Let $\mathbf{F}(x,y) = \langle 3+2xy^2, 2x^2y\rangle$. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is the arc of the hyperbola $\cancel{y = \frac{1}{x}}$ from $(1,1)$ to $\left(4, \frac{1}{4}\right)$.

$\vec{F}$ is conservative, $\int_C F\cdot dr = \int_C \nabla f\cdot dr$

$= f(4, \frac{1}{4}) - f(1,1)$

$= 13 - 4 = \boxed{9}$

Example 8: Given $\mathbf{F}(x,y) = \langle \underset{P}{3x^2 - 4y}, \underset{Q}{4y^2 - 2x} \rangle$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the curve given by

$\boxed{\mathbf{r}(t) = \langle t^3 + 1, t^2 \rangle,\ 0 \le t \le 1.}$

Step 1 Check to see if F is conservative

$\frac{\partial Q}{\partial x} = -2, \quad \frac{\partial P}{\partial y} = -4$ not conservative. Path dependent curve matters.

$$\int_C F \cdot dr = \int_a^b F(\vec{r(t)}) \cdot r'(t)\,dt$$

$$= \int_0^1 \langle 3(t^3+1) - 4t^2, 4t^4 - 2(t^3+1) \rangle \cdot \langle 3t^2, 2t \rangle\,dt$$

$$= \int_0^1 \left(3(t^3+1) - 4t^2\right)3t^2 + \left(4t^4 - 2(t^3+1)\right)(2t)\,dt$$

Example 9: Given that $\mathbf{F} = \langle 4xe^z, \cos(y), 2x^2e^z \rangle$ is conservative and $\mathbf{r}(t) = \langle \sin(t), t, \cos(t) \rangle$, compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ for $0 \le t \le \frac{\pi}{2}$. Note: We had to tell you $\mathbf{F}$ is conservative since we have not yet learned the testing criteria for conservativness in $\mathbb{R}^3$

given F is conservative $F = \nabla f$

Find $f(x,y,z)$

$$\int 4xe^z\,dx = 2x^2e^z + g(y,z)$$

$$\int \cos y\,dy = \sin y + h(x,z)$$

$$\int 2x^2e^z\,dz = 2x^2e^z + k(x,y)$$

$$\boxed{f(x,y,z) = 2x^2e^z + \sin y}$$

$r(t) = \langle \sin t, t, \cos t \rangle$, $0 \le t \le \frac{\pi}{2}$

$r(\frac{\pi}{2}) = (1, \frac{\pi}{2}, 0)$

$r(0) = (0, 0, 1)$

$$\int_C F \cdot dr = \int_C \nabla f \cdot dr$$

$$= f\left(r\left(\tfrac{\pi}{2}\right)\right) - f(r(0))$$

$$= f(1, \tfrac{\pi}{2}, 0) - f(0,0,1)$$

$$= 2 + 1 - 0$$

$$= 3$$