Section 16.4 Greens Theorem

We learned in section 16.2 how to find a line integral $\int_C f(x,y)dx + g(x,y)dy$ over a curve C parameterized by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$.

Example 1: Evaluate $\int_C y^2 dx + x dy$, where C: $\mathbf{r}(t) = \left\langle t^2, 2t \right\rangle$, $0 \leq t \leq 2$.

In this section, we will learn how to compute a line integral over a **closed curve**.

Defintion: A **closed curve** is a curve in which its terminal point coincides with its initial point. A **simple closed curve** is a closed curve that does not cross itself anywhere between its endpoints.

Note: If we are computing a line integral over a closed curve, we write $\oint_C$ instead of $\int_C$.

Green's Theorem: Let C be a positively oriented (counterclockwise) piecewise-smooth simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA$$

This says that the line integral over a simple closed curve C is equal to a double integral over the area of the region D the curve C encloses.

Note: We only use Green's theorem if we are on a **positively oriented closed** curve. If the curve is not positively oriented, then change the sign of the line integral. If not explicitly stated, assume counterclockwise orientation.

Example 2: Evaluate $\oint_C x^3 dx + xy dy$ where C is the triangular path from $(0,0)$ to $(1,1)$ to $(0,1)$ then back to $(0,0)$. Assume positive orientation.

Example 3: Evaluate $\oint_C (y^2 + \cos x)dx + (x - \arctan y)dy$ where C is the curve that encloses the region bounded by $y = 4 - x^2$ and $y = 0$. Assume positive orientation.

Example 4: Evaluate $\oint_C x^2 y dx + y^3 dy$ where C is the line segment from $(-1, 0)$ to $(1, 0)$ and then around the top half of the circle $x^2 + y^2 = 1$.

What does Green's theorem look like if $\mathbf{r}(t)$ is in a vector field? Suppose we are asked to find $\int_c \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{r}(t)$ is a closed curve.

Let $\mathbf{F} = \langle P, Q\rangle$ be a vector field and let C be a **simple closed curve** defined by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, $a \le t \le b$.

Then $d\mathbf{r} = \mathbf{r}'(t) = \langle x'(t)dt, y'(t)dt\rangle = \langle dx, dy\rangle$, and therefore $\mathbf{F}\cdot d\mathbf{r} = \langle P, Q\rangle \cdot \langle dx, dy\rangle = Pdx + Qdy$. Thus if we are on a simple closed curve, Green's theorem gives

$$\int_c \mathbf{F}\cdot d\mathbf{r} = \oint_C Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\, dA$$

Question: What is the work done if we are moving a particle along a **closed** curve, $\mathbf{r}(t)$, in a **conservative** vector field?

Example 5: Suppose a particle travels one revolution clockwise around the unit circle under the force field $\mathbf{F}(x, y) = \left\langle e^x - y^3, \cos(y) + x^3\right\rangle$. Find the work done .