Section 16.4 Greens Theorem

We learned in section 16.2 how to find a line integral $\int_C f(x,y)dx + g(x,y)dy$ over a curve C parameterized by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$.

Example 1: Evaluate $\int_C y^2 dx + x dy$, where C: $\mathbf{r}(t) = \langle t^2, 2t \rangle$, $0 \le t \le 2$.

$x = t^2 \quad dx = 2t\,dt$

$y = 2t \quad dy = 2dt$

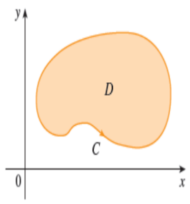
$$\int_C y^2 dx + x\,dy = \int_0^2 4t^2\, 2\,dt + t^2\, 2t\,dt$$

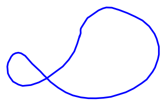
$$= \int_0^2 (8t^2 + 2t^3)\,dt = \boxed{\frac{112}{3}}$$

In this section, we will learn how to compute a line integral over a **closed curve**.

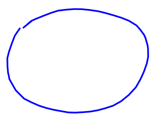
Defintion: A **closed curve** is a curve in which its terminal point coincides with its initial point. A **simple closed curve** is a closed curve that does not cross itself anywhere between its endpoints.

Note: If we are computing a line integral over a closed curve, we write $\oint_C$ instead of $\int_C$.





not simple



is simple

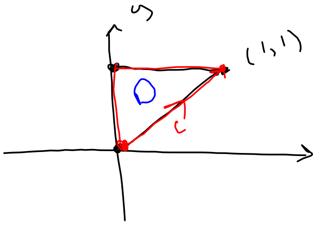
Green's Theorem: Let C be a positively oriented (counterclockwise) piecewise-smooth simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

This says that the line integral over a simple closed curve C is equal to a double integral over the area of the region D the curve C encloses.

Note: We only use Green's theorem if we are on a **positively oriented closed** curve. If the curve is not positively oriented, then change the sign of the line integral. If not explicitly stated, assume counterclockwise orientation.

Example 2: Evaluate $\oint_C x^3 dx + \underline{xy} dy$ where C is the triangular path from $(0,0)$ to $(1,1)$ to $(0,1)$ then back to $(0,0)$. Assume positive orientation.



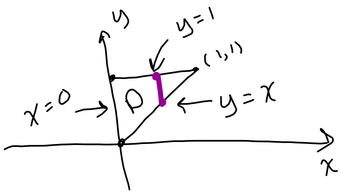
① closed curve C thus C encloses a region D in the xy-plane.

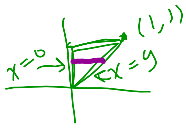
$Q = xy$

$P = x^3$

② By Green's theorem,

$$\oint_C x^3 dx + xy\, dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$





$$= \iint_D (y - 0)\, dA$$

$$= \iint_D y\, dA$$

$D: \begin{cases} 0 \le x \le 1 \\ x \le y \le 1 \end{cases}$

or

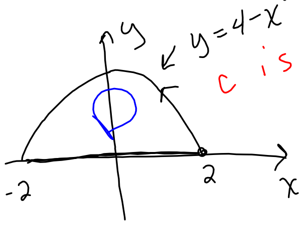
$\boxed{D: \begin{array}{l} 0 \le y \le 1 \\ 0 \le x \le y \end{array}}$

$$= \int_0^1 \int_0^y y\, dx\, dy$$

$$= \int_0^1 \left(yx \Big|_{x=0}^{x=y} \right) dy$$

$$= \int_0^1 (y^2\, dy) = \frac{y^3}{3}\Big|_0^1 = \boxed{\frac{1}{3}}$$

Example 3: Evaluate $\oint_C \overset{P}{(y^2+\cos x)}\,dx + \overset{Q}{(x-\arctan y)}\,dy$ where C is the curve that encloses the region bounded by $y = 4 - x^2$ and $y = 0$. Assume positive orientation.



c is closed & counterclockwise

use Green's!

by Green's

$$= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \iint_D (1 - 2y)\, dA$$

D: $-2 \le x \le 2$

$0 \le y \le 4 - x^2$

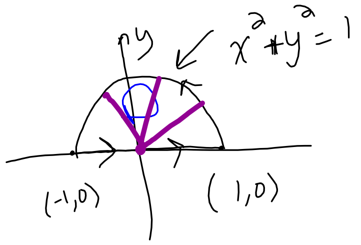
$$= \int_{-2}^{2} \int_0^{4-x^2} (1-2y)\, dy\, dx$$

$$= \int_{-2}^{2} \left[(y - y^2) \Big|_{y=0}^{y=4-x^2} \right] dx$$

$$= \int_{-2}^{2} \left[4 - x^2 - \underbrace{(4-x^2)^2}_{\text{Foil}} \right] dx$$

$$= \frac{-352}{15}$$

Example 4: Evaluate $\oint_C \overset{P}{x^2y}\,dx + \overset{Q}{y^3}\,dy$ where C is the line segment from $(-1,0)$ to $(1,0)$ and then around the top half of the circle $x^2+y^2=1$.



$$\oint_C x^2y\,dx + y^3\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint_D (0 - x^2)\, dA$$

$$= -\iint_D x^2\, dA$$

$$= -\int_0^{\pi}\int_0^1 r^2\cos^2\theta\, r\,dr\,d\theta$$

$$= -\int_0^{\pi}\cos^2\theta\,d\theta \int_0^1 r^3\,dr$$

$$= -\int_0^{\pi}\frac{1}{2}(1+\cos 2\theta)\,d\theta \int_0^1 r^3\,dr$$

$$= -\left[\frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\Bigg|_0^{\pi} \frac{r^4}{4}\Bigg|_0^1\right]$$

$$= -\left[\frac{1}{2}(\pi)\frac{1}{4}\right]$$

$$= -\frac{\pi}{2}$$

D is polar

$0 \le r \le 1$

$0 \le \theta \le \pi$

$x = r\cos\theta$

$dA = r\,dr\,d\theta$

Fubini!

What does Green's theorem look like if $\mathbf{r}(t)$ is in a vector field? Suppose we are asked to find $\int_c \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{r}(t)$ is a closed curve.

Let $\mathbf{F} = \langle P, Q\rangle$ be a vector field and let C be a **simple closed curve** defined by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, $a \le t \le b$. Then $d\mathbf{r} = \mathbf{r}'(t) = \langle x'(t)dt, y'(t)dt\rangle = \langle dx, dy\rangle$, and therefore $\mathbf{F}\cdot d\mathbf{r} = \langle P, Q\rangle \cdot \langle dx, dy\rangle = P dx + Q dy$. Thus if we are on a simple closed curve, Green's theorem gives

$$\int_c \mathbf{F}\cdot d\mathbf{r} = \oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Question: What is the work done if we are moving a particle along a **closed** curve, $\mathbf{r}(t)$, in a **conservative** vector field?

misconception: If C is a closed curve, then $W = \int_C F\cdot dr = 0$ not always true!!

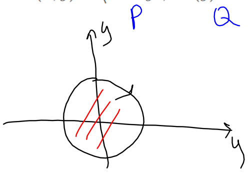
If F is conservative & C is closed,

$$\int_C F\cdot dr = \int_C \nabla f\cdot dr = f(r(b)) - f(r(a)) = 0$$

c closed means $r(b) = r(a)$

Example 5: Suppose a particle travels one revolution clockwise around the unit circle under the force field $\mathbf{F}(x, y) = \langle e^x - y^3, \cos(y) + x^3\rangle$. Find the work done .

P Q



is F conservative?

does $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$? no! not conservative!

$3x^2 = -3y^2$

$W = \int_C F\cdot dr$ is c closed? yes!

is c counterclockwise? no

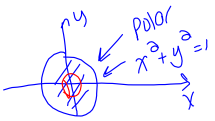
reverse orientation $\to -C$

By Green's theorem

$P = e^x - y^3$

$Q = \cos y + x^3$

$$W = \int_C F\cdot dr = -\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$



$0 \le r \le 1$

$0 \le \theta \le 2\pi$

$$= -\int_0^{2\pi}\int_0^1 \left[3x^2 - (-3y^2)\right] dA$$

$$= -\int_0^{2\pi}\int_0^1 3(x^2+y^2)\, dA$$

$$= -\int_0^{2\pi}\int_0^1 3r^2\, r\, dr\, d\theta$$

$$= -\int_0^{2\pi} d\theta \int_0^1 3r^3\, dr$$

$$= -\left[\theta\Big|_0^{2\pi} \frac{3r^4}{4}\Big|_0^1\right]$$

$$= -2\pi\cdot\frac{3}{4}$$

$$= -\frac{3\pi}{2}$$