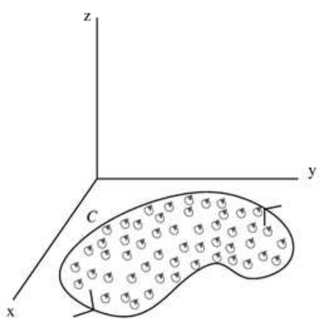
Section 16.8/16.9 Stokes' Theorem and The Divergence Theorem

Recall Surface Integrals over vector fields: Let $\mathbf{F}$ be a vector field whose domain includes the positively oriented surface S, where S is defined parametrically by $\mathbf{r}(u, v)$, $u, v \in D$. Then the surface integral of $\mathbf{F}$ over S, also called the **Flux** of $\mathbf{F}$ over S, is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

Recall Green's Theorem: Let $F = \langle P, Q \rangle$ be a vector field and let C be a positively oriented, piecewise-smooth, simple closed curve in the plane that encloses a region D. Then

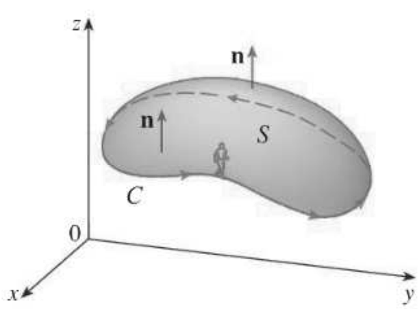
$$\int_c \mathbf{F} \cdot d\mathbf{r} = \oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$



Note: We can only use Green's Theorem if the curve C lies in a plane. Stokes' Theorem allows us to compute a line integral over a closed curve *in space*.

Stokes' Theorem: Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive (counterclockwise) orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\Re^3$ that contains S.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$$



This means the work done in moving a particle through a vector field along a closed curve is mathematically equivalent to the flux of the curl $\mathbf{F}$ over any surface the curve encloses.

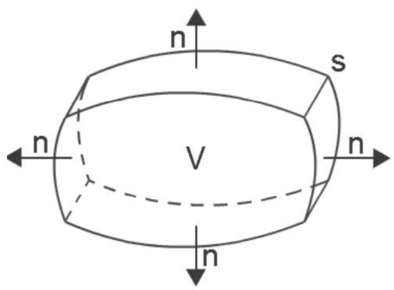
Example 1: Use Stokes' Theorem to find $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \langle z^2, 2x, y^2 \rangle$ and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$. Orient C to be counterclockwise when looking from above (which ensures the normal vector points upward).

Example 2: Use Stokes' Theorem to find $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \langle z^2, y^2, xy \rangle$ where C is the triangle with vertices $(1,0,0)$, $(0,1,0)$, $(0,0,2)$. Orient C to be counterclockwise when looking from above.

Example 3: Use Stokes' Theorem to find $\iint\limits_S \text{curl}\, \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = \left\langle x^2 \sin z, y^2, xy \right\rangle$ and S is the part of the paraboliod $z = 1 - x^2 - y^2$ that lies above the xy plane, oriented upward.

A surface integral over a **closed surface** can be evaluated as a triple integral over the volume enclosed by the surface.

Divergence Theorem Let E be a simple solid region whose boundary surface has positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then



$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div } \mathbf{F} dV$$

Example 4: Use the Divergence Theorem to evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = \langle x + \sin z, 2y + \cos x, 3z + \tan y \rangle$ over the sphere $x^2 + y^2 + z^2 = 4$.

Example 5: Let S be the surface of the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane. Use the Divergence Theorem to evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \left\langle x^3, 2xz^2, 3y^2z \right\rangle$.

Example 6: Using the Divergence Theorem, find the flux of the vector field $\mathbf{F} = \langle z\cos y, x\sin z, xz \rangle$ where S is the tetrahendron bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $2x + y + z = 2$.