16.8 16.9

Section 16.8/16.9 Stokes' Theorem and The Divergence Theorem

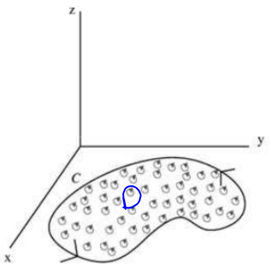
Recall Surface Integrals over vector fields: Let $\mathbf{F}$ be a vector field whose domain includes the positively oriented surface S, where S is defined parametrically by $\mathbf{r}(u,v)$, $u,v \in D$. Then the surface integral of $\mathbf{F}$ over S, also called the **Flux** of $\mathbf{F}$ over S, is

check orientation!

16.7
$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

Recall Green's Theorem: Let $F = \langle P, Q\rangle$ be a vector field and let C be a positively oriented, piecewise-smooth, simple closed curve in the plane that encloses a region D. Then

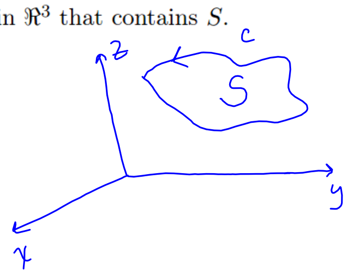
$$\int_C \mathbf{F}\cdot d\mathbf{r} = \oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\, dA$$

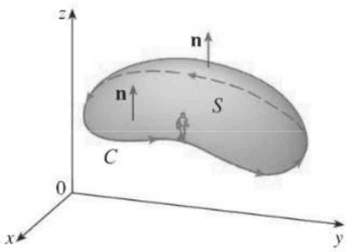


Note: We can only use Green's Theorem if the curve C lies in a plane. Stokes' Theorem allows us to compute a line integral over a closed curve *in space*.

Stokes' Theorem: Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive (counterclockwise) orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\Re^3$ that contains S.

$$\boxed{\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F}\cdot d\mathbf{S}}$$



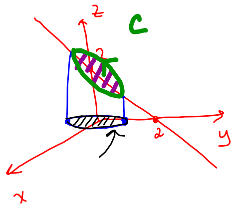


This means the work done in moving a particle through a vector field along a closed curve is mathematically equivalent to the flux of the curl $\mathbf{F}$ over any surface the curve encloses.

Example 1: Use Stokes' Theorem to find $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F} = \langle z^2, 2x, y^2\rangle$ and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$. Orient C to be counterclockwise when looking from above (which ensures the normal vector points upward).

$\boxed{\text{Stokes' } \int_C F\cdot dr = \iint_S \text{curl} F\cdot dS,}$ S = surface C encloses.

Step 1: Parameterize the surface that C encloses



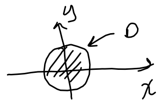
$y+z=2$

C encloses the surface $\boxed{y+z=2}$ ← parameterize

let $x=x$, $y=y$, $z=2-y$

$z=2-y$

S ↓ $\boxed{r(x,y) = \langle x, y, 2-y\rangle, \quad x^2+y^2\le 1}$



D: $x^2+y^2\le 1$

Step 2: Find $\text{curl} F = \nabla\times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z^2 & 2x & y^2 \end{vmatrix}$

$\boxed{\text{curl} F = \langle 2y, 2z, 2\rangle}$

$\int_C F\cdot dr = \iint_S \text{curl} F\cdot dS$ $\quad S: r(x,y) = \langle x, y, 2-y\rangle,$ $\quad D: x^2+y^2\le 1$

check orientation

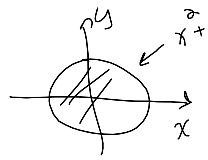
$\overset{16.7}{=} \iint_D \text{curl}F(r(x,y))\cdot(r_x\times r_y)\,dA$

$r_x\times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{vmatrix}$

$= \langle 0, 1, 1\rangle$ correct orientation $k>0$

$= \iint_{x^2+y^2\le 1} \langle 2y, 2(2-y), 2\rangle\cdot\langle 0,1,1\rangle\,dA$

$= \iint_{x^2+y^2\le 1} (2(2-y)+2)\,dA = \iint_{x^2+y^2\le 1} (6-2y)\,dA$



$x^2+y^2\le 1$

$0\le r\le 1$
$0\le\theta\le 2\pi$
$y = r\sin\theta$
$dA = r\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^1 (6-2r\sin\theta)\,r\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^1 (6r-2r^2\sin\theta)\,dr\,d\theta$

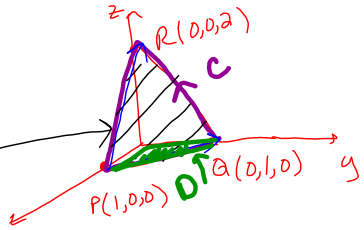
$= \int_0^{2\pi} \left(3r^2 - \frac{2}{3}r^3\sin\theta\right)\Big|_{r=0}^{r=1} d\theta$

$= \int_0^{2\pi}\left(3-\frac{2}{3}\sin\theta\right)d\theta$

$= \boxed{6\pi}$

Example 2: Use Stokes' Theorem to find $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F} = \langle z^2, y^2, xy\rangle$ where C is the triangle with vertices $(1,0,0)$, $(0,1,0)$, $(0,0,2)$. Orient C to be counterclockwise when looking from above.

Stokes: $\int_{\boxed{C}} F\cdot dr = \iint_{\boxed{S}} \text{curl}F\cdot dS$



C = boundary curve of S

Step 1 Parameterize the surface C encloses. S is the plane.

equation of a plane is $\vec{n}\cdot(\vec{r}-\vec{r}_0)=0$ $\quad \vec{n} = PQ\times PR$

$$= \begin{vmatrix} i & j & k \\ -1 & 1 & 0 \\ -1 & 0 & 2 \end{vmatrix}$$

$\langle 2,2,1\rangle\cdot\langle x-1, y, z-0\rangle = 0$

$2x-2+2y+z=0$

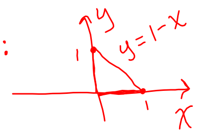
$\boxed{z=2-2x-2y}$

$\vec{n} = \langle 2,2,1\rangle$

$x=x,\ y=y,\ z=2-2x-2y$

$\boxed{\begin{array}{l} r(x,y)=\langle x, y, 2-2x-2y\rangle \\ D:\quad 0\le x\le 1,\quad 0\le y\le 1-x \end{array}}$

D:



Step 2: $\text{curl}F = \nabla\times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z^2 & y^2 & xy \end{vmatrix} = \langle x, -(y-2z), 0\rangle$

$\boxed{\text{curl}F = \langle x, 2z-y, 0\rangle}$

$r_x\times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & -2 \\ 0 & 1 & -2 \end{vmatrix} = \langle 2,2,1\rangle$ good orientati

$$\int_C F\cdot dr = \iint_S \text{curl}F\cdot dS = \iint_{\substack{0\le x\le 1\\ 0\le y\le 1-x}} \text{curl}F(r(x,y))\cdot(r_x\times r_y)\,dA$$

$$= \iint_{\substack{0\le x\le 1\\ 0\le y\le 1-x}} \langle x, \underbrace{2(2-2x-2y)-y}_{4-4x-4y-y}, 0\rangle\cdot\langle 2,2,1\rangle\,dA$$

$$= \iint_{\substack{0\le x\le 1\\ 0\le y\le 1-x}} \langle x, -5y-4x+4, 0\rangle\cdot\langle 2,2,1\rangle\,dA$$

$$= \iint (2x-10y-8x+8)\,dA$$

$$= \iint_{\substack{0\le x\le 1\\ 0\le y\le 1-x}} (-6x-10y+8)\,dA$$

$$= \int_0^1\int_0^{1-x} (-6x-10y+8)\,dy\,dx$$

correct set-up! for $dA = dy\,dx$

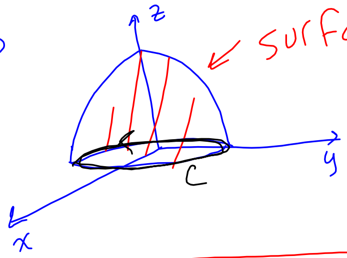
Example 3: Use Stokes' Theorem to find $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = \langle x^2 \sin z, y^2, xy \rangle$ and S is the part of the paraboliod $z = 1 - x^2 - y^2$ that lies above the xy plane, oriented upward.

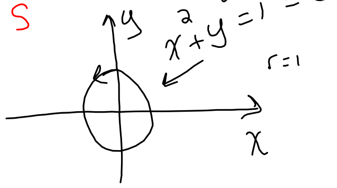
STOKES' theorem ① $\int_C F \cdot dr = \iint_S \text{curl} F \cdot dS$

② $\iint_S \text{curl} F \cdot dS = \int_C F \cdot dr$

step 1: parameterize the boundary curve of C $\quad r(t) = \langle x(t), y(t), z(t) \rangle, \; a \le t \le a$

what is S?





$\boxed{C: \; r(t) = \langle \cos t, \sin t, 0 \rangle,}$ $\quad 0 \le t \le 2\pi$

$F = \langle x^2 \sin z, y^2, xy \rangle$

stokes

$\iint_S \text{curl} F \cdot dS = \int_C F \cdot dr$

$= \int_0^{2\pi} F(r(t)) \cdot r'(t)\, dt$

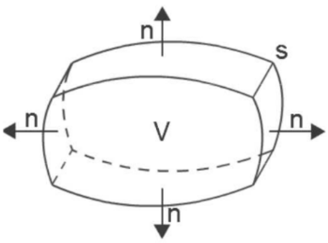
$= \int_0^{2\pi} \langle 0, \sin^2 t, \cos t \sin t \rangle \cdot \langle -\sin t, \cos t, 0 \rangle\, dt$

$= \int_0^{2\pi} (\cos t \sin^2 t)\, dt$ $\quad u = \sin t$ ($t = 2\pi, u = 0$; $t = 0, u = 0$), $du = \cos t\, dt$

$= \int_0^0 u^2\, du = \boxed{0}$

A surface integral over a **closed surface** can be evaluated as a triple integral over the volume enclosed by the surface.

Divergence Theorem Let E be a simple solid region whose boundary surface has positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then



$$\iint_{S} \mathbf{F} \cdot d\mathbf{S} = \iiint_{E} \text{div } \mathbf{F} dV$$

Example 4: Use the Divergence Theorem to evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = \langle x + \sin z, 2y + \cos x, 3z + \tan y \rangle$ over the sphere $x^2 + y^2 + z^2 = 4$.

Divergence theorem says: $\iint_S F \cdot dS = \iiint_E \operatorname{div} F \, dV$

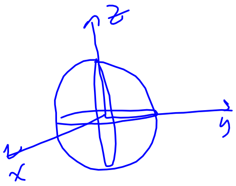
① $\boxed{\text{Find } \operatorname{div}(F) = 6}$

$$\nabla \cdot \vec{F} = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle \cdot \langle x + \sin z, 2y + \cos x, 3z + \tan y \rangle$$

$$= 1 + 2 + 3$$

$$= 6$$

② Define E: interior of the sphere $x^2 + y^2 + z^2 = 4$



$0 \le \rho \le 2$

$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi$

in spherical, $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$$\iint_S F \cdot dS = \iiint_E 6 \, dV$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^{2} 6 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi \, d\phi \int_0^{2} 6\rho^2 \, d\rho$$

$$= \left(\theta \Big|_0^{2\pi} \right) \left(-\cos\phi \Big|_0^{\pi} \right) \left(2\rho^3 \right) \Big|_0^{2}$$

$$= (2\pi)(2)(16)$$

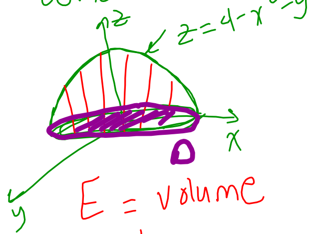
$$= \boxed{64\pi}$$

Example 5: Let S be the surface of the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane. Use the Divergence Theorem to evaluate $\boxed{\iint_S \mathbf{F} \cdot d\mathbf{S}}$, where $\mathbf{F} = \langle x^3, 2xz^2, 3y^2z \rangle$.

① $\operatorname{div} F = \nabla \cdot F = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle \cdot \langle x^3, 2xz^2, 3y^2 z \rangle$

$$\boxed{\operatorname{div} F = 3x^2 + 3y^2}$$

what is S?



Divergence theorem: $\iint_S F \cdot ds = \iiint_E \operatorname{div} F \, dV$

E = volume S encloses.

$$= \iiint_E (3x^2 + 3y^2)\, dV$$

$$= \iint_D \left[\int_0^{4-x^2-y^2} \underbrace{(3x^2+3y^2)}_{3r^2} dz \right] dA$$

$4-x^2-y^2 \to 4-r^2$

$dA \to r\,dr\,d\theta$

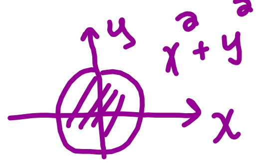
$0 \le r \le 2$

$0 \le \theta \le 2\pi$

define E:

$0 \le z \le 4 - x^2 - y^2$

$x^2 + y^2 \le 4$



$0 \le r \le 2$

$0 \le \theta \le 2\pi$

$0 \le z \le 4 - r^2$

$$= \int_0^{2\pi} \int_0^2 \int_0^{4-r^2} 3r^2 \, dz \, r\,dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^2 \int_0^{4-r^2} 3r^3 \, dz\,dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^2 3r^3 z \Big|_{z=0}^{z=4-r^2} dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^2 3r^3(4-r^2)\, dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^2 (12r^3 - 3r^5)\, dr$$

$$= \theta \Big|_0^{2\pi} \left[3r^4 - \frac{1}{2} r^6 \right] \Big|_0^2$$

$$\boxed{= (2\pi)\left(3(16) - \frac{64}{2}\right)}$$

Example 6: Using the Divergence Theorem, find the flux of the vector field $\mathbf{F} = \langle z\cos y, x\sin z, xz\rangle$ where S is the tetrahendron bounded by the planes $x=0$, $y=0$, $z=0$, and $2x+y+z=2$.

Recall: $\text{Flux } F = \iint_S F\cdot dS = \boxed{\iiint_E \operatorname{div} F\, dV}$

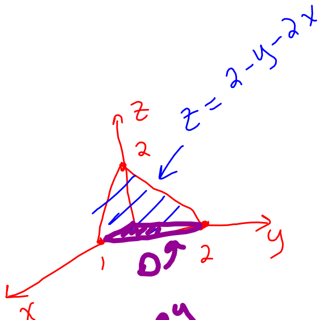
$$\operatorname{div} F = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z\rangle \cdot \langle z\cos y, x\sin z, xz\rangle$$

$$\operatorname{div}(F) = 0+0+x$$

$\operatorname{div} F = x$

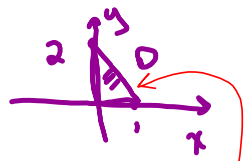
define S:

what is E



$$0 \le z \le 2-y-2x$$

$$\boxed{\begin{array}{l} 0 \le y \le 2-2x \\ 0 \le x \le 1 \end{array}} \; D$$



$0 = 2-y-2x$

$\boxed{y = 2-2x}$

$$\text{Flux } F = \boxed{\iiint_E x\, dV}$$

$$= \iint_D \left[\int_0^{2-y-2x} x\, dz\right] dA$$

$$= \int_0^1\int_0^{2-2x}\int_0^{2-y-2x} x\, dz\, dy\, dx$$

$$= \int_0^1\int_0^{2-2x} xz\Big|_{z=0}^{z=2-y-2x} dy\, dx$$

$$= \int_0^1\int_0^{2-2x} x(2-y-2x)\, dy\, dx$$

$$= \int_0^1\int_0^{2-2x} \left(2x - xy - 2x^2\right) dy\, dx$$

$$= \int_0^1 \left(2xy - \frac{xy^2}{2} - 2x^2 y\right)\Big|_{y=0}^{y=2-2x} dx$$

$$= \int_0^1 \left[2x(2-2x) - \frac{x}{2}(2-2x)^2 - 2x^2(2-2x)\right] dx$$