Section 16.5 Curl and Divergence

In this section, we define two operations on vector fields. These operations are called **Divergence** and **Curl**, which are characteristics of how fluid/flow is behaving in a small neighborhood around a given point.

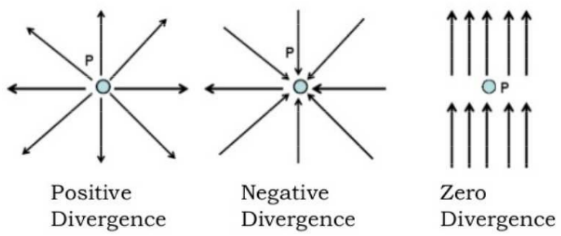
Definition: Divergence is a measurement of how much fluid/flow enters the neighborhood around a point P compared to how much fluid/flow exits the neighborhood around P.

If more fluid/flow enters the neighborhood around P than leaves the neighborhood around P, the divergence will be **negative**(gaining fluid/flow in that neighborhood).

If the same amount of fluid/flow enters the neighborhood around P as leaving it, the divergence will be **zero**.

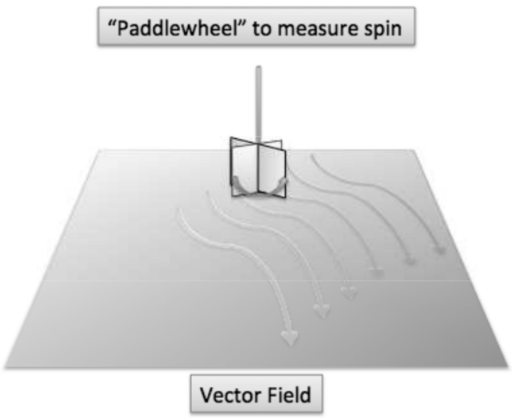
If more fluid/flow leaves the neighborhood around P than enters the neighborhood around P, the divergence will be **positive** (losing fluid in that neighborhood). In this case, we say the vector field is divergent at the point P.

Illustration of the divergence of a vector field at point P:



Definition: The curl of $\mathbf{F}$ measures the tendency of the fluid/flow to rotate in a vector field of a neighborhood around that point.

Think of circulation as being the amount of pushing, twisting, or turning around the point P. We can visualize curl by the paddle wheel illustration shown below.



Definition: The **del operator**, denoted by ∇, is defined as $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$. Note: this is **not** the same as the gradient!!

Definition: Divergence and Curl If $\mathbf{F} = \langle P, Q, R \rangle$ is a a vector field on $\Re^3$ and the partial derivatives of P, Q, and R all exist, then

a.) The **divergence of** F is $\nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$.

∇ is not the gradient

b.) The **curl of F** is the vector field on $\Re^3$ defined by curl $\mathbf{F} = \nabla \times \mathbf{F}$.

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, -\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right), \_\_\_ \right\rangle$$

Example 1: Find the divergence of $\mathbf{F}(x, y, z) = \langle x^2 y, yz^2, zx^2 \rangle$ at the point $(1, -1, 1)$, $(1, 1, 1)$, and $(1, -5, 1)$. Interpret your answer.

$$\text{div } F = \nabla \cdot F = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle x^2 y, yz^2, zx^2 \rangle$$

$$\text{div } F = 2xy + z^2 + x^2$$

$$\text{div } F(1,-1,1) = -2+1+1 = 0$$

same amount entering as leaving around that point

$$\text{div } F(1,1,1) = 2+1+1 = 4 > 0$$

more leaving as entering

$$\text{div } F(1,-5,1) = -10+1+1 = -8 < 0$$

more entering as leaving

Example 2: Find the divergence and curl of $\mathbf{F} = \langle xy, xz, xyz^2 \rangle$.

$$\text{div } F = \nabla \cdot F = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle xy, xz, xyz^2 \rangle$$

$$= y + 0 + 2xyz$$

$$\text{curl } F = \nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & xz & xyz^2 \end{vmatrix}$$

$$= \langle xz^2 - x, -(yz^2 - 0), z - x \rangle$$

$$= \langle xz^2 - x, -yz^2, z - x \rangle$$

Theorem: If **F** is a vector field defined on all of $\Re^3$ whose component functions have continuous partial derivatives and curl $\mathbf{F} = \mathbf{0}$, then **F** is a conservative vector field. This gives us a way to determine whether a vector function on $\Re^3$ is conservative.

For recall purposes, the 'conservative test' on $\Re^2$: $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle = P\mathbf{i} + Q\mathbf{j}$ is a conservative vector field if and only if $\frac{\partial Q}{dx} = \frac{\partial P}{dy}$

Example 5: Determine if the vector field is conservative. $\mathbf{F} = \langle y^2z^3,\ 2xyz^3,\ 4xy + z\rangle$. If it is conservative, find the potential function f.

$$\text{curl } F = \nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2z^3 & 2xyz^3 & 4xy+z \end{vmatrix}$$

must be $\langle 0,0,0\rangle$ for F to be conservative!

$$= \langle \underline{4x - 6xyz^2},\ \text{who cares},\ \text{who cares} \rangle$$

not 0! F not conservative

Example 6: If $\mathbf{F} = \langle x,\ e^y \sin z,\ e^y \cos z\rangle$, Find $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{r}(t) = \langle t^4, t, 2t^2\rangle$, for $1 \le t \le 2$

check for conservative!

$$\text{curl } F = \nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & e^y \sin z & e^y \cos z \end{vmatrix}$$

$$= \langle e^y \cos z - e^y \cos z,\ 0-0,\ 0-0 \rangle$$

$$= \langle 0,0,0\rangle \quad \text{conservative!}$$

Example 6: If $\mathbf{F} = \langle x,\ e^y \sin z,\ e^y \cos z\rangle$, Find $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{r}(t) = \langle t^4, t, 2t^2\rangle$, for $\boxed{1 \le t \le 2}$

F conservative means $F = \nabla f$, f = potential function

$$\int_C F\cdot dr = \int_C \nabla f \cdot dr$$

$r(2) = \langle 16, 2, 8\rangle$

$r(1) = \langle 1, 1, 2\rangle$

$$= f(r(2)) - f(r(1))$$

$$= f(16,2,8) - f(1,1,2)$$

Find f

$\int x\,dx,\quad \int e^y \sin z\,dy,\quad \int e^y \cos z\,dz$

$\downarrow \qquad \downarrow \qquad \downarrow$

$\frac{x^2}{2} \qquad e^y \sin z \qquad e^y \sin z$

$$f(x,y,z) = \frac{x^2}{2} + e^y \sin z$$

$$= \frac{16^2}{2} + e^2 \sin 8 - \left(\frac{1}{2} + e \sin 2\right)$$