Recap of key concepts

$\underset{\text{curve}}{C}$ parameterized by $r(t)$, $a \le t \le b$

$$\int_C \underbrace{f(x,y)}_{\text{scalar function}} ds = \int_a^b f(r(t))|r'(t)|\, dt$$

$$\int_C \underbrace{F}_{\text{vector function}} \cdot dr = \int_C F(r(t)) \cdot r'(t)\, dt$$

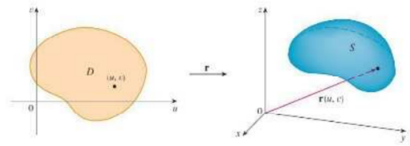
f = potential function

if F is conservative $[\, F = \nabla f \,]$

$$\int_C F \cdot dr = \int_C \nabla f\, dr = f(r(b)) - f(r(a))$$

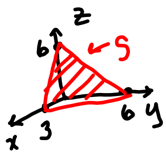
Section 16.6 Parametric Surfaces and their Areas

In this section, we will learn how to find the area of a surface defined parametrically. Recall how we defined a space curve by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$. Notice here, only *one* parameter was necessary, namely t. In much the same way, we will describe a **surface** by a vector function $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$. Note in order to parameterize a surface, we need **two** parameters, u and v. As u and v vary over the domain D, $\mathbf{r}(u,v)$ traces out the surface as terminal points of the position vector $\mathbf{r}(u,v)$.



Example 1: Parameterize the following surfaces

a.) $z + 2x + y = 6$



$z = 6 - 2x - y$

let $x = x$

$y = y$

$z = 6 - 2x - y$

$$\boxed{r(x,y) = \langle x, y, 6 - 2x - y \rangle}$$

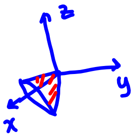
b.) $-5y - 4z + x = 20$

$x = 20 + 4z + 5y$

$r(y,z) = \langle 20 + 4z + 5y, y, z \rangle$

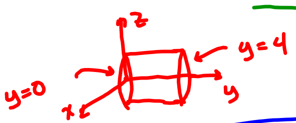
c.) $x = \sqrt{y^2 + z^2}$

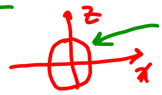
cone



$r(y,z) = \langle \sqrt{y^2+z^2}, y, z \rangle$

d.) The part of the cylinder $x^2 + z^2 = 1$ that lies between the planes $y = 0$ and $y = 4$.

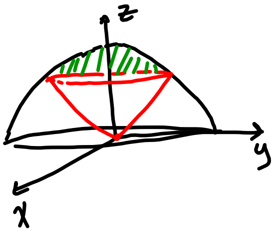




circle is parameterized by $x = \cos\theta$, $z = \sin\theta$, $y = y$

$$\boxed{S = r(y,\theta) = \langle \cos\theta, y, \sin\theta \rangle \quad 0 \le y \le 4}$$

e.) The part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the cone $z = \sqrt{x^2 + y^2}$.



Parameterize a sphere

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

$\rho = 2$

$r(\theta,\phi) = \langle 2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi \rangle$

$0 \le \theta \le 2\pi, \quad 0 \le \phi \le \frac{\pi}{4}$

since we are above the cone

$z = \sqrt{x^2+y^2}$

$z^2 = x^2 + y^2$

$x^2 + y^2 + z^2 = 4$

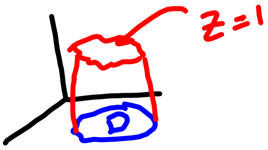
$2z^2 = 4 \quad z = \sqrt{2}$

plug $z = \sqrt{2}$ into $z = \rho\cos\phi$

$\sqrt{2} = 2\cos\phi$

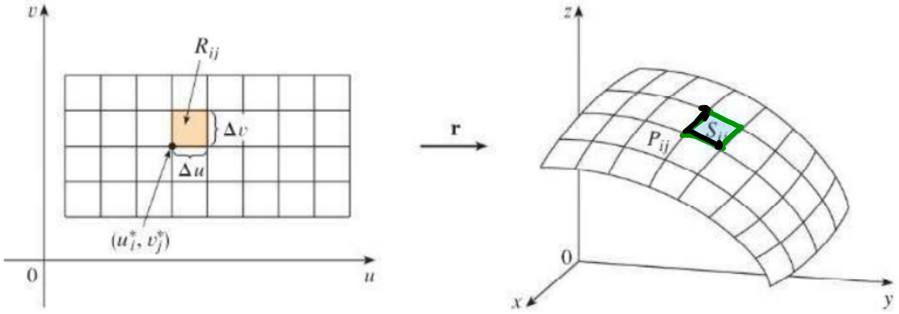
$\phi = \frac{\pi}{4}$

Recall from section 15.2, we learned how to find the area of a region D that lies in one of the coordinate planes using a double integral:

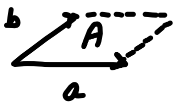


$$A(D) = \iint_D 1\, dA$$

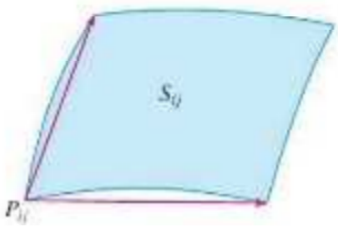
Suppose now we wish to find the (surface) area of a region that does not lie in a coordinate plane. Partition D into a set of rectangles R_{ij}. Choose (u_i^*, v_j^*) be the lower left corner of R_{ij}. We call S_{ij} a patch and has position vector $\mathbf{r}(u_i^*, v_j^*)$ as one of its corners.



Notice by the figure below how the edges of the patch can be approximated by vectors. These vectors in turn can be approximated by the tangent vectors $\mathbf{r}_u^*(u_i^*, v_j^*)$ and $\mathbf{r}_v^*(u_i^*, v_j^*)$.



$A = |a \times b|$

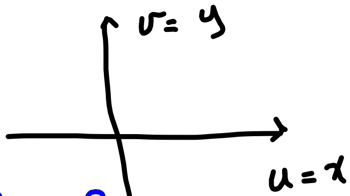


Recall the tangent plane is useful in approximating surfaces, and we also know the area of the parallelogram determined by **a** and **b** is $|\mathbf{a} \times \mathbf{b}|$. Thus we can approximate the area of the patch with the parallelogram determined by the tangent vectors.

Definition: If a smooth parametric surface S is given by $\mathbf{r}(u, v)$, and S is covered just once as (u, v) ranges throughout the parametric domain D, then the **surface area** of S is

D= Parametric doman

$$\boxed{A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA}$$



r(u,v)= Parameterization of the surface, S.

Example 2: Find the surface area of the part of the plane $\underbrace{4x+2y+z=8}_{S}$ that lies in the first octant.

① parameterize S

$z = 8-2y-4x$

$r(x,y) = \langle x, y, 8-2y-4x \rangle$

$r_x = \langle 1, 0, -4 \rangle$

$r_y = \langle 0, 1, -2 \rangle$

$$\begin{vmatrix} i & j & k \\ 1 & 0 & -4 \\ 0 & 1 & -2 \end{vmatrix}$$

$\langle 4, 2, 1 \rangle$

② $A(S) = \iint_D |r_x \times r_y| \, dA$

$= \iint_D |\langle 4,2,1\rangle| \, dA$

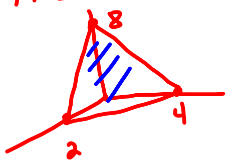
$= \iint_D \sqrt{21} \, dA$

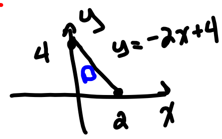
$= \int_0^2 \int_0^{-2x+4} \sqrt{21} \, dy\,dx$

D = domain of $r(x,y)$

$z = 8-2y-4x$

First octant





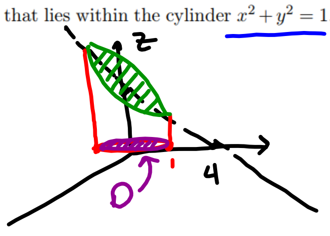
$\sqrt{21} \int_0^2 y \Big|_{y=0}^{y=-2x+4} dx$

$\sqrt{21} \int_0^2 (-2x+4)\, dx = \sqrt{21} \left[-x^2+4x \right] \Big|_0^2$

$= \sqrt{21}(4)$

Example 3: Find the surface area of the part of the plane $\underline{z=4-y}$ that lies within the cylinder $\underline{x^2+y^2=1}$.

① $r(x,y) = \langle x, y, 4-y \rangle \leftarrow$



② $A(S) = |r_x \times r_y|$

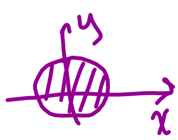
$$r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{vmatrix} = \langle 0, 1, 1 \rangle$$

$A(S) = \iint_D \sqrt{2} \, dA$

look at D

$0 \le r \le 1$

$0 \le \theta \le 2\pi$



$= \int_0^{2\pi} \int_0^1 \sqrt{2} \, r\,dr\,d\theta$

$= \sqrt{2} \int_0^{2\pi} d\theta \int_0^1 r\,dr = (\sqrt{2})(2\pi)\left(\frac{1}{2}\right)$

$= \sqrt{2}\pi$

inside

Example 4: Find the area of the part of the surface $y = x^2 + z^2$ that lies ~~inside~~ the cylinder $x^2 + z^2 = 2$.

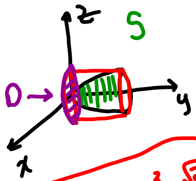
① $r(x,z) = \langle x, x^2+z^2, z \rangle$

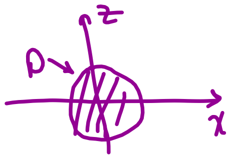
$$r_x \times r_z = \begin{vmatrix} i & j & k \\ 1 & 2x & 0 \\ 0 & 2z & 1 \end{vmatrix} = \langle 2x, -1, 2z \rangle$$

$$A(S) = \iint_D |r_x \times r_z|\, dA = \iint_D \sqrt{4x^2+1+4z^2}\, dA$$

$$\to \int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{4r^2+1}\; r\, dr\, d\theta$$

$y = x^2+z^2$ inside $x^2+z^2=2$





$$\int_0^{2\pi} d\theta \int_0^{\sqrt{2}} r\sqrt{4r^2+1}\, dr$$

u-sub

$$\theta\Big|_0^{2\pi} \frac{1}{8}\frac{2}{3}(4r^2+1)^{\frac{3}{2}}\Big|_0^{\sqrt{2}}$$

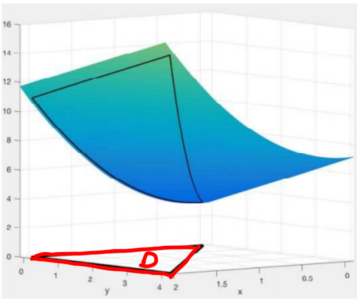
$$(2\pi)\frac{1}{12}\left[9^{3/2}-1\right]$$

$0 \le \theta \le 2\pi$, $0 \le r \le \sqrt{2}$

in polar, $x^2+z^2=r^2$

$\to$ Answer $= \frac{\pi}{6}\left(9^{3/2}-1\right)$

Example 5: Find the surface area of the part of the surface $z = 2x^2 + y + 3$ that lies above the triangle with vertices $(0, 0)$, $(2, 0)$ and $(2, 4)$.

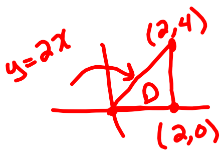


$r(x,y) = \langle x, y, 2x^2+y+3 \rangle$

$$r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & 4x \\ 0 & 1 & 1 \end{vmatrix}$$

$$= \langle -4x, -1, 1 \rangle$$

$$A(S) = \iint_D |r_x \times r_y|\, dA = \iint_D \sqrt{16x^2+2}\, dA$$



$$= \int_0^2\int_0^{2x} \sqrt{16x^2+2}\; dy\, dx$$

$$= \int_0^2 \underline{2x\sqrt{16x^2+2}}\, dx$$

u-sub!

$$= \frac{2}{32}\,\frac{2}{3}(16x^2+2)^{3/2}\Big|_0^2$$

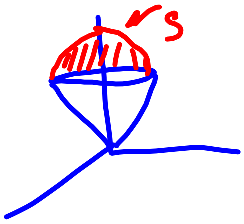
$$= \frac{1}{24}\left[(16(4)+2)^{3/2} - 2^{3/2}\right]$$

Recall from spherical coordinates, we can parameterize a sphere as $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$ and $z = \rho\cos\phi$. Thus $r(\theta, \phi) = \langle \rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi\rangle$, and

$$\boxed{\begin{aligned} \mathbf{r}_\phi \times \mathbf{r}_\theta &= \langle \rho^2 \sin^2\phi\cos\theta, \rho^2\sin^2\phi\sin\theta, \rho^2\sin\phi\cos\phi\rangle \\ \text{and} \quad |\mathbf{r}_\theta \times \mathbf{r}_\phi| &= \rho^2\sin(\phi) \end{aligned}}$$

will be given on the exam.

Example 6: Find the surface area of the part of the sphere $x^2 + y^2 + z^2 = 16$ that lies above the cone $z = \sqrt{x^2 + y^2}$.



$\rho = 4$, so $A(S) = \iint 16 \sin\phi \, dA$

$0 \le \theta \le 2\pi$

$0 \le \phi \le \frac{\pi}{4}$

$z^2 = x^2 + y^2 \to 2z^2 = 16$
$\to z = 2\sqrt{2}$

$z = \rho\cos\phi$

$2\sqrt{2} = 4\cos\phi$

$\frac{\sqrt{2}}{2} = \cos\phi$

$\phi = \frac{\pi}{4}$

$$A(S) = \int_0^{2\pi}\int_0^{\frac{\pi}{4}} 16\sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} 16\sin\phi \, d\phi$$

$$= \theta\Big|_0^{2\pi} \left(-16\cos\phi\right)\Big|_0^{\frac{\pi}{4}}$$

$$= (2\pi)\left(-16\left[\frac{\sqrt{2}}{2} - 1\right]\right)$$