## Section 16.7 Surface Integrals

Recall: If a smooth parametric surface $S$ is given by the equation $\mathbf{r}(u, v)$ and $S$ is covered just once as $(u, v)$ ranges throughout the parametric domain $D$, then the **surface area** of $S$ is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA$$

**Definition:** Now suppose we want to integrate a function $f(x, y, z)$ over a surface $S$ defined by the equation $\mathbf{r}(u, v)$ and $S$ is covered just once as $(u, v)$ ranges throughout the parametric domain $D$, then the **surface integral of $f$ over $S$ is**

$$\iint_S f(x, y, z) \, dS = \iint_D f(\mathbf{r}(u, v)) |\mathbf{r}_u \times \mathbf{r}_v| \, dA$$

Example 1: Evaluate $\iint_S (x + y + z) \, dS$ where $S$ is defined by $\mathbf{r}(u, v) = \langle u + v, u - v, 1 + u + 2v \rangle$, $0 \le u \le 1$ and $0 \le v \le 2$.

$$\iint_S f(x,y,z)\, dS = \iint_{\boxed{D}} f(r(u,v)) |r_u \times r_v| \boxed{dA}$$

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix} = \langle 3, -1, -2 \rangle$$

$$\int_0^1 \int_0^2 (u + v + u - v + 1 + u + 2v)\sqrt{14}\, dv\, du$$

$$\int_0^1 \int_0^2 (3u + 2v + 1)\sqrt{14}\, dv\, du$$

Example 1b: Set up but do not evaluate $\iint_S y \, dS$ where $S$ is the part of the plane $2x + y + z = 4$ that lies in the first octant.

$\iint_S f(x,y,z)\, dS = \iint_D f(r(u,v)) |r_u \times r_v|$, where

$r(u,v)$ is the parameterization of $S$

and $D$ = domain of $r(u,v)$.

step 1: Parameterize $2x + y + z = 4$

$z = 4 - y - 2x$

let $x = x$, $y = y$, $z = 4 - y - 2x$

$\boxed{r(x,y) = \langle x, y, 4 - y - 2x \rangle}$

$$r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & -2 \\ 0 & 1 & -1 \end{vmatrix} = \langle 2, 1, 1 \rangle$$

$|r_x \times r_y| = \sqrt{6}$

$\iint_S y\, dS$

$\iint_D y\sqrt{6}\, dA$

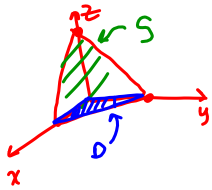
what is $D$?

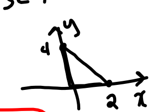
$D$ = domain of surface $z = 4 - y - 2x$

$D$ in $xy$ plane, set $z = 0$

$0 = 4 - y - 2x$

$y = 4 - 2x$




$$= \boxed{\int_0^2 \int_0^{4-2x} y\sqrt{6}\, dy\, dx}$$

Example 2: Evaluate $\iint_S (x+2y+z)\,dS$ where $S$ is the part of the plane $y+z=4$ that is inside the cylinder $x^2+y^2=1$.

$$\iint_S f(x,y,z)\,dS = \iint_D f(r(u,v))\,|r_u \times r_v|\,dA$$

$\boxed{z = 4-y}$

$x = x$
$y = y$
$z = 4-y$

$$r(x,y) = \langle x, y, 4-y \rangle$$

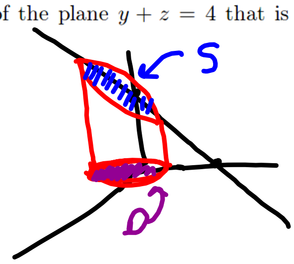
$$r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{vmatrix} = \langle 0, 1, 1 \rangle$$

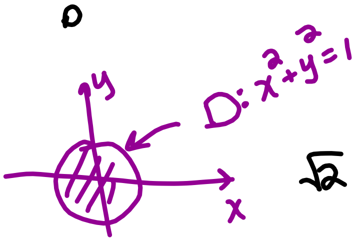
$$|r_x \times r_y| = \sqrt{2}$$

$$\iint_D f(r(x,y))\,|r_x \times r_y|\,dA$$

Example 2: Evaluate $\iint_S (x+2y+z)\,dS$ where $S$ is the part of the plane $y+z=4$ that is inside the cylinder $\boxed{x^2+y^2=1}$

$$\iint_D (x+2y+4-y)\sqrt{2}\,dA$$




$$\sqrt{2}\iint_D (x+y+4)\,dA$$

$0 \le \theta \le 2\pi$
$0 \le r \le 1$

$$\sqrt{2}\int_0^{2\pi}\int_0^1 (r\cos\theta + r\sin\theta + 4)\,r\,dr\,d\theta$$

$$\sqrt{2}\int_0^{2\pi}\int_0^1 (r^2\cos\theta + r^2\sin\theta + 4r)\,dr\,d\theta$$

$$\frac{1}{3}r^3\cos\theta + \frac{1}{3}r^3\sin\theta + 2r^2\Big|_0^1$$

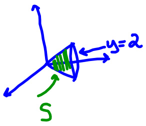
$$\sqrt{2}\int_0^{2\pi}\left(\frac{1}{3}\cos\theta + \frac{1}{3}\sin\theta + 2\right)d\theta$$

$$\sqrt{2}\left(\frac{1}{3}\sin\theta - \frac{1}{3}\cos\theta + 2\theta\right)\Big|_0^{2\pi}$$

$$\sqrt{2}\left[0 - \frac{1}{3}[1-1] + 4\pi\right] = \boxed{4\sqrt{2}\pi}$$

Example 3: Evaluate $\iint_S yz^2\,dS$ where $S$ is the part of the cone $y=\sqrt{x^2+z^2}$, with $0\le y\le 2$.

$$\iint_S f(x,y,z)\,dS = \iint_D f(r(u,v))\,|r_u\times r_v|\,dA$$




let $x=x$, $z=z$, $y=\sqrt{x^2+z^2}$

$r(x,z)=\langle x, \sqrt{x^2+z^2}, z\rangle$

$\frac{1}{2}(x^2+z^2)^{-\frac{1}{2}}(2x)$

$$r_x\times r_z = \begin{vmatrix} i & j & k \\ 1 & \frac{x}{\sqrt{x^2+z^2}} & 0 \\ 0 & \frac{z}{\sqrt{x^2+z^2}} & 1 \end{vmatrix}$$

$$=\left\langle \frac{x}{\sqrt{x^2+z^2}}, -1, \frac{z}{\sqrt{x^2+z^2}}\right\rangle$$

$$|r_x\times r_z| = \sqrt{\frac{x^2}{x^2+z^2}+1+\frac{z^2}{x^2+z^2}} = \sqrt{1+1} = \sqrt{2}$$

$\frac{x^2+z^2}{x^2+z^2}=1$

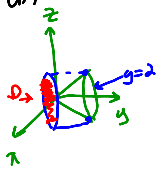
$$\boxed{|r_x\times r_z| = \sqrt{2}}$$

$r(x,z)=\langle x, \sqrt{x^2+z^2}, z\rangle$

Example 3: Evaluate $\iint_S yz^2\,dS$ where $S$ is the part of the cone $y=\sqrt{x^2+z^2}$, with $0\le y\le 2$.

$$\iint_S yz^2\,dS = \iint_D r(x,z)\,|r_x\times r_z|\,dA$$

$$=\iint_D \sqrt{x^2+z^2}\,z^2\sqrt{2}\,dA$$

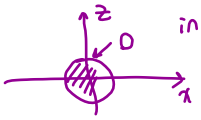



$y=\sqrt{x^2+z^2}$, $y=2$

D is the intersection of these two surfaces, projected onto the zx plane

$2=\sqrt{x^2+z^2}$

$4=x^2+z^2$




in polar,

$0\le\theta\le 2\pi$

$0\le r\le 2$

$x^2+z^2=r^2$

$z=r\sin\theta$

$$\sqrt{2}\iint_D \sqrt{x^2+z^2}\,z^2\,dA$$

$$\sqrt{2}\int_0^{2\pi}\int_0^2 (r)(r\sin\theta)^2\,r\,dr\,d\theta$$

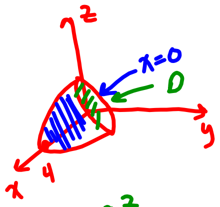
$$\sqrt{2}\int_0^{2\pi}\int_0^2 r^4\sin^2\theta\,dr\,d\theta$$

$$\sqrt{2}\int_0^{2\pi}\frac{1}{2}(1-\cos 2\theta)\,d\theta\int_0^2 r^4\,dr$$

$$\sqrt{2}\left(\frac{1}{2}\left(\theta-\frac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi}\right)\frac{r^5}{5}\Big|_0^2$$

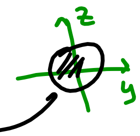
$$\sqrt{2}\left(\cancel{\tfrac{1}{2}}(\cancel{2}\pi)\right)\frac{32}{5} = \boxed{\frac{32}{5}\sqrt{2}\,\pi}$$

Example 4: Set up but do not evaluate $\iint_S (y^2+z^2)\,dS$ where $S$ is part of the paraboloid $x = 4 - y^2 - z^2$ that lies in front of the plane $x = 0$.




$r(y,z) = \langle 4-y^2-z^2,\ y,\ z \rangle$

$$r_y \times r_z = \begin{vmatrix} i & j & k \\ -2y & 1 & 0 \\ -2z & 0 & 1 \end{vmatrix} = \langle 1, 2y, 2z \rangle$$




$0 = 4 - y^2 - z^2$

$y^2 + z^2 = 4$

$$|r_y \times r_z| = \sqrt{1+4y^2+4z^2} = \sqrt{1+4(y^2+z^2)} = \sqrt{1+4r^2}$$

$$\iint_S (y^2+z^2)\,dS = \int_0^{2\pi}\int_0^2 r^2\sqrt{1+4r^2}\ r\,dr\,d\theta$$

Recall from spherical coordinates, we can parameterize a sphere as $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$ and $z = \rho\cos\phi$. Thus $r(\theta,\phi) = \langle \rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \cancel{\rho\cos\phi}\rangle$. Then

$$\boxed{\mathbf{r}_\phi \times \mathbf{r}_\theta = \langle \rho^2\sin^2\phi\cos\theta, \rho^2\sin^2\phi\sin\theta, \rho^2\sin\phi\cos\phi\rangle}$$

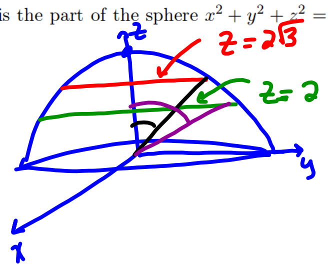
and

$$|\mathbf{r}_\theta \times \mathbf{r}_\phi| = \rho^2\sin(\phi)$$

Example 5: Evaluate $\iint_S z\,dS$, where $S$ is the part of the sphere $x^2+y^2+z^2 = 16$ that lies between the planes $z = 2$ and $z = 2\sqrt{3}$

here, $\boxed{\rho = 4}$

$\boxed{0 \le \theta \le 2\pi}$




$z = \rho\cos\phi$

• $2\sqrt{3} = 4\cos\phi$

$\frac{\sqrt{3}}{2} = \cos\phi$

$\phi = \frac{\pi}{6}$

• $2 = 4\cos\phi$

$\frac{1}{2} = \cos\phi$

$\phi = \frac{\pi}{3}$

$\frac{\pi}{6} \le \phi \le \frac{\pi}{3}$

$$\iint_S z\,dS = \int_0^{2\pi}\int_{\pi/6}^{\pi/3} (4\cos\phi)\underbrace{(16\sin\phi)}_{\rho^2\sin\phi}\,d\phi\,d\theta$$

$$64\int_0^{2\pi} d\theta \int_{\pi/6}^{\pi/3} \cos\phi\sin\phi\,d\phi$$

u-sub

$u = \sin\phi$

$du = \cos\phi\,d\phi$

$\int u\,du$

$\frac{u^2}{2}$

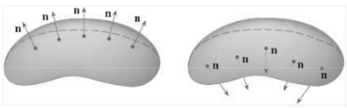
$$(64)\left(\theta\Big|_0^{2\pi}\right)\left(\frac{\sin^2\phi}{2}\Bigg|_{\pi/6}^{\pi/3}\right)$$

$$(64)(2\pi)\left(\frac{\left(\frac{\sqrt{3}}{2}\right)^2}{2} - \frac{\left(\frac{1}{2}\right)^2}{2}\right)$$

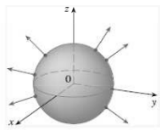
**Surface Integrals through Vector Fields**

Here, we study surfaces in vector fields. Since surfaces are two sided, we must have an orientation of the surface. The normal vectors to the surface provide the orientation, but since there are two normal vectors to a surface, we need a convention. The usual orientation we choose is the upward orientation. If the problem does not state the orientation, assume positive (upward). If the problem states upward orientation, we want the normal to have a positive **k** component. If the problem states downward orientation, we want the normal to have a negative **k** component. For closed surfaces, the convention is to have the normal vectors that point outward from the surface, called positive orientation.

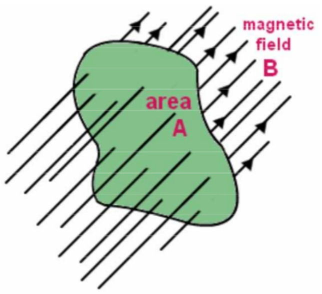
The two orientations of an orientable surface



Positive orientation



Definition: Suppose $\mathbf{F} = \langle P, Q, R\rangle$ is a vector field that contains surface $S$. The amount of flow that passes through the surface $S$ is called the **Flux** of **F** across the surface $S$. In other words, the flux of **F** describes the flow at any point across the surface. If $\mathbf{n}$ is a unit normal vector to the surface at any point on $S$, notice that $\mathbf{n}$ 'passes through' the surface $S$.




Definition: Let $F = \langle P, Q, R\rangle$ be a vector field whose domain includes a surface $S$ where $S$ is defined parametrically by $\mathbf{r}(u, v)$. Then the **Flux** of **F** over $S$ is

$$\boxed{\iint_S \mathbf{F}\cdot d\mathbf{S}} = \iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \boxed{\iint_D \mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u \times \mathbf{r}_v)\, dA}$$

Example 6: Evaluate $\iint_S \mathbf{F}\cdot d\mathbf{S}$ where $\mathbf{F} = \langle y, 2z, x\rangle$ and $S$ is defined by $\mathbf{r}(u,v) = \langle 3u+v, u-2v, 3-u+v\rangle$ and $0 \le u \le 1$, $0 \le v \le 1$. Assume positive (upward) orientation.

$$\iint_S F\cdot dS = \iint_D F(r(u,v))\cdot(r_u \times r_v)\,dA$$

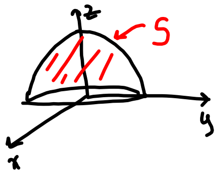
$$r_u \times r_v = \begin{vmatrix} i & j & k \\ 3 & 1 & -1 \\ 1 & -2 & 1 \end{vmatrix} = \langle -1,-4,-7\rangle \to \langle 1,4,7\rangle$$

$k$ must be positive!

$$\int_0^1\int_0^1 \langle u-2v,\ 2(3-u+v),\ 3u+v\rangle \cdot \langle 1,4,7\rangle\,dA$$

$$\int_0^1\int_0^1 (14u+13v+24)\,du\,dv = \boxed{\frac{75}{2}}$$

Example 7: Find the flux of $\mathbf{F} = \langle x, y, z\rangle$ where $S$ is part of the surface $z = 1 - x^2 - y^2$ above the $xy$-plane. Assume positive (upward) orientation.




$$\text{Flux } F = \iint_S F\cdot dS$$

Parameterize $S$:

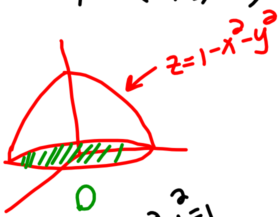
$$r(x,y) = \langle x, y, 1-x^2-y^2\rangle$$

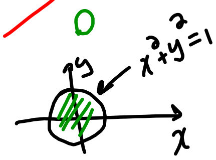
$$r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} = \langle 2x, 2y, 1\rangle$$

good orientation since $k > 0$

$$\iint_S F\cdot dS = \iint_D F(r(x,y))\cdot(r_x \times r_y)\,dA$$

$\vec{F} = \langle x, y, z\rangle$, so $\iint_D \langle x, y, 1-x^2-y^2\rangle\cdot\langle 2x, 2y, 1\rangle\,dA$




$$\iint_D (2x^2+2y^2+1-x^2-y^2)\,dA$$

$$\iint_D (x^2+y^2+1)\,dA$$

$$\int_0^{2\pi}\int_0^1 (r^2+1)\,r\,dr\,d\theta$$

$$\int_0^{2\pi} d\theta \int_0^1 (r^3+r)\,dr$$

$$\theta\Big|_0^{2\pi}\left(\frac{1}{4}r^4+\frac{r^2}{2}\right)\Big|_0^1$$

$$\boxed{\frac{3\pi}{2}}$$

Section 16.7 (continued)

Recall: Let $S$ be a surface parameterized by $r(u,v)$, where $D$ is the domain of $r(u,v)$.

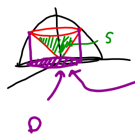
Then

① $\iint_S f(x,y,z)\,dS = \iint_D f(r(u,v))\,|r_u \times r_v|\,dA$

② $\iint_S F\cdot dS = \iint_D F(r(u,v))\cdot(r_u \times r_v)\,dA$

③ Flux $F = \iint_S F\cdot dS$

Example 8: Find the flux of the vector field $\mathbf{F} = \langle x, y, -2x\rangle$ across the surface $S$, where $S$ is the part of the cone $z = \sqrt{x^2+y^2}$ that lies ~~below~~ inside the sphere $x^2+y^2+z^2 = 8$. Assume positive (upward) orientation.




Flux $F = \iint_S F\cdot dS$

intersection of $z = \sqrt{x^2+y^2}$, $x^2+y^2+z^2=8$

$z^2 = x^2+y^2$ 

$2z^2 = 8$

$z^2 = 4$

$\boxed{z=2}$

$\boxed{4 = x^2+y^2}$

$S$ = cone

Parameterize the cone

$z = \sqrt{x^2+y^2}$

$r(x,y) = \langle x, y, \sqrt{x^2+y^2}\rangle$

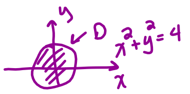
Flux $F = \iint_S F\cdot dS = \iint_D F(r(x,y))\cdot(r_x \times r_y)\,dA$

$$r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & \frac{x}{\sqrt{x^2+y^2}} \\ 0 & 1 & \frac{y}{\sqrt{x^2+y^2}} \end{vmatrix} = \left\langle \frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}, 1\right\rangle$$

$\vec{F} = \langle x, y, -2x\rangle \leftarrow \boxed{r(x,y) = \langle x, y, \sqrt{x^2+y^2}\rangle}$

$$\iint_D F(r(x,y))\cdot(r_x \times r_y)\,dA = \iint_D \langle x,y,-2x\rangle\cdot\left\langle \frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}, 1\right\rangle dA$$

$$= \iint_D \left(\frac{-x^2}{\sqrt{x^2+y^2}} - \frac{y^2}{\sqrt{x^2+y^2}} - 2x\right) dA$$




$$= \iint_D \left(\frac{-x^2-y^2}{\sqrt{x^2+y^2}} - 2x\right) dA$$

$$= -\iint_D \left(\frac{x^2+y^2}{\sqrt{x^2+y^2}} + 2x\right) dA$$

$$= -\int_0^{2\pi}\int_0^2 \left(\frac{r^2}{r} + 2r\cos\theta\right) r\,dr\,d\theta$$

$$= -\int_0^{2\pi}\int_0^2 \left(r^2 + 2r^2\cos\theta\right) dr\,d\theta$$

$$= -\int_0^{2\pi} \left(\frac{r^3}{3} + \frac{2r^3}{3}\cos\theta\right)\Bigg|_{r=0}^{r=2} d\theta$$

$$= -\int_0^{2\pi} \left(\frac{8}{3} + \frac{16}{3}\cos\theta\right) d\theta$$

$$= -\left(\frac{8}{3}\theta + \frac{16}{3}\sin\theta\right)\Bigg|_0^{2\pi}$$

$$= \boxed{-\frac{8}{3}(2\pi)}$$