Section 5.1

1. Given the graph of $f(x)$ find intervals if increasing/decreasing. local extrema, intervals of concavity and inflection points. Given that $f(x)$ is continuous and $f(0) = 0$, sketch a possible graph of $f(x)$.

- If $f'>0$ then $f$ is increasing.
- If $f'<0$ then $f$ is decreasing.
- If $f''$ goes from positive to negative at $x=a$ then $f$ has a local max at $x=a$.
- If $f''$ goes from negative to positive at $x=a$ then $f$ has a local minimum at $x=a$.

- $f$ inc on $(\frac{1}{2}, 2)$ and $(10, 12)$
- $f$ dec on $(-\frac{1}{2}, \frac{1}{2})$ and $(8, 10)$
- local min at $x = \frac{1}{2}, x = 10$
- local max at $x = 8$

- If $f'' > 0$ then $f''$ is increasing then $f$ is concave up.
- If $f'' < 0$ then $f''$ is decreasing then $f$ is concave down.

- $f$ concave up: $(-\frac{1}{2}, 2.5) U (4.5, 6.2)$
  $U (9, 12)$
- $f$ concave down: $(2.5, 4.5) U (4.2, 9)$

- Inflection points: $x = 2.5, 4.5, 6.2$.
2. Sketch a graph satisfying:
   a.) Domain: All real numbers
   b.) \( f(-1) = -2 \), \( f(0) = 0 \), \( f(2) = 3 \)
   c.) \( f'(x) < 0 \) for \( x < -1 \) and \( x > 2 \)
   d.) \( f'(x) > 0 \) if \( -1 < x < 2 \)
   e.) \( f''(x) > 0 \) if \( x < 0 \) and \( f''(x) < 0 \) if \( x > 0 \)

\[ \text{c.)} \quad \text{d.)} \]
\[ \text{min} \quad \text{max} \]
\[ \text{e.)} \]
\[ \text{CU} \quad \text{CD} \]
Inflexion point @ \( x=0 \)
Section 5.2

3. For the following functions, identify all critical values.

a.) \( f(x) = 4x^3 - 9x^2 - 12x + 3 \)

\[
f'(x) = 12x^2 - 18x - 12
\]

\[
= 6(2x^2 - 3x - 2)
\]

\[
= 6(2x + 1)(x - 2)
\]

\( f'(x) = 0 \) if \( x = -\frac{1}{2}, x = 2 \)

---

b.) \( f(x) = x^2 e^{2x} \)

\[
f'(x) = 2x e^{2x} + x^2 2e^{2x}
\]

\[
= 2xe^{2x}(1 + x)
\]

\( f'(x) = 0 \) if \( 2x = 0 \)

\( x = 0 \)

---

c.) \( f(x) = |x^2 - 2x| \)

This is wrong:

\[
f'(x) = 2x - 2\]

\( f'(1) = 0 \)

\( f'(0) = 0 \)

\( f'(2) = 0 \)

---

Def. A critical value is a value of \( c \) in the domain of \( f \) such that either

1. \( f'(c) = 0 \)
2. \( f'(c) \) does not exist
d.) \( f(x) = (x^2 - x)^{1/3} \)

\[
\begin{align*}
\frac{d}{dx} f(x) &= \frac{1}{3} \left( x^2 - x \right)^{-\frac{3}{3}} (2x - 1) \\
&= \frac{2x - 1}{3 \left[ x(x-1) \right]^{\frac{3}{3}}} \\
&= \frac{2x - 1}{3x(x-1)}
\end{align*}
\]

\( c \land: \ x = \frac{1}{2}, \ f'(x) = 0 \)

\( x = 0, 1 \) \( f' \) dne

e.) \( f(x) = \frac{x + 1}{x - 2} \)

\[
\begin{align*}
\frac{d}{dx} f(x) &= \frac{(1)(x-2) - (x+1)(1)}{(x-2)^2} \\
&= \frac{x - 2 - x - 1}{(x-2)^2} \\
&= \frac{-3}{(x-2)^2}
\end{align*}
\]

\( \text{no critical numbers} \)
4. Find the absolute and local extrema for the following functions by graphing:

a) \( f(x) = x - x^2 \), \(-1 < x < 2\)
   - Local max at \((0,1)\)
   - Local min \(x = \frac{1}{2}\)
   - Absolute max = 1
   - Absolute min = -3

b) \( f(x) = \frac{x^2}{2} - x^3 \) if \(-1 < x < 0\) or \(0 < x < 1\)

5. Find the absolute extrema for:

a) \( f(x) = x^3 - 12x + 1 \) over the interval \([-1,3]\)
   - If \( f(x) \) is continuous on \([0,6]\) the absolute extrema will occur either at a critical number or an endpoint.
   - Find all \( c \) that \( f'(c) = 0 \) or \( c \) is in the given interval
   - \( f(x) = x^3 - 12x + 1 \)
   - \( f'(x) = 3x^2 - 12 \)
   - \( f'(x) = 3(x^2 - 4) \)
   - \( x = 2, x = -2 \)
   - **Absolute max** = 6
   - **Absolute min** = -15

b) \( f(x) = \ln(x) + x \) over the interval \([1,3]\)
   - Evaluate \( f(\ln(1)) = f(0) = 0 \)
   - Evaluate \( f(1) = f(1) = 1 \)
   - Evaluate \( f(3) = 3 \ln(3) \)
   - **Absolute max** = ***not in [1,3]***
6. Sketch a graph of a function satisfying the following conditions:
   
a.) \( x = 2 \) is a critical number, but \( f(x) \) has no local extrema.

   ![Graph of a function with a critical point at \( x = 2 \) but no local extrema.]

b.) \( f(x) \) is a continuous function with a local maximum at \( x = 2 \), but \( f(x) \) is not differentiable at \( x = 2 \).

   ![Graph of a function with a local maximum at \( x = 2 \) but not differentiable at that point.]

   ![Graph of a function with a local maximum at \( x = 2 \) and a vertical tangent at that point.]

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Section 5.3

7. State the Mean Value Theorem. Verify \( f(x) = x^2 \) satisfies the Mean Value Theorem on the interval \((-1, 2)\). Find all \( c \) that satisfies the conclusion of the Mean Value Theorem.

**MVT: If \( f(x) \) is continuous on \([a, b]\) and differentiable on \((a, b)\), then there is a solution \( c \) to \( f'(c) = \frac{f(b) - f(a)}{b - a} \).**

Then, there is a solution to \( f'(c) = \frac{f(2) - f(-1)}{2 - (-1)} = \frac{4 - 1}{3} = 1 \).

Solve \( f'(x) = 1 \):

\[
2x = 1
\]

\[
x = \frac{1}{2}
\]

8. Find the intervals where the given function is increasing or decreasing and identify all local extrema:

a) \( f(x) = 3x^4 + 4x^3 - 12x^2 + 8 \)

Look at the sign of \( f'(x) \):

\[
f'(x) = 12x^3 + 12x^2 - 24x
\]

\[
= 12x(x^2 + x - 2)
\]

\[
= 12x(x + 2)(x - 1)
\]

Local min at \(( -2, f(-2)) \) and \(( 1, f(1)) \):

\[
\text{min} \quad (-2, -24) \quad (1, 3)
\]

Local max at \(( 0, f(0)) \) :

\[
\text{max} \quad (0, 8)
\]
b) \( y = \tan^{-1}(x^2) = \arctan(x^2) \)

\[
y' = \frac{1}{1 + x^4} (2x)
\]

\[\frac{d}{dx} \arctan x = \frac{1}{1+x^2}\]

Critical point at \( x = 0 \)

Local min at \((0,0)\)
c) \( f(x) = \frac{x}{(x-1)^2} \)

Vertical asymptote at \( x=1 \).

\[
f'(x) = \frac{(1)(x-1)^2 - x \cdot 2(x-1)}{(x-1)^4} = \frac{(x-1)(x-1-2x)}{(x-1)^4} = \frac{-x^2 - 2x}{(x-1)^3}
\]

Critical points:

- \((-\infty, -1)\) and \((1, +\infty)\)

Dec: \((-\infty, -1)\) and \((1, +\infty)\)

Inc: \((-1, 1)\)

No local max

Local min at \((-1, f(-1))\)

\((-1, -\frac{1}{4})\)
For $f(x) = x \sin x + \cos x$ on $[0, 2\pi]$

$\frac{d}{dx} f(x) = (1) \sin x + x \cos x + -\sin x$

$f'(x) = x \cos x$

$f'(x) = 0$ when $x \cos x = 0$

$x = 0, \frac{\pi}{2}, \frac{3\pi}{2}$

increasing $(0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi)$

decreasing $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

max $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$, min $\left(\frac{3\pi}{2}, -\frac{3\pi}{2}\right)$
9. Determine the intervals where the given function is concave up or concave down and identify all inflection points for \( f(x) = x^4 + 5x^2 \).

\[
\begin{align*}
\text{sign of } f''(x), & \quad f'(x) = 5x^4 + 20x^3 \\
& \quad f''(x) = 20x^3 + 60x^2 \\
& \quad f''(x) = 0 \text{ when } x = 0, x = -3 \\
\text{CD: } (-\infty, -3), \quad \text{CU: } (-3, 0) \cup (0, \infty) \\
\text{point of inflection is } & \quad (-3, f(-3)) = (-3, 16.2)
\end{align*}
\]

Second derivative test for local extrema: Let \( x = c \) be a critical number of \( f(x) \).

\[
\begin{align*}
1 & \quad \text{if } f''(c) > 0 \quad \text{local minimum at } (c, f(c)) \\
2 & \quad \text{if } f''(c) < 0 \quad \text{local maximum at } (c, f(c)) \\
3 & \quad \text{if } f''(c) = 0 \text{ or dne} \quad \text{test fails}
\end{align*}
\]

10. Given \( f(-3) = 4, f'(-3) = 0, f''(-3) = 7, f(2) = -5, f'(2) = 0, \) and \( f''(2) = -6 \), identify any local extrema of \( f \).

\[
\begin{align*}
\text{CN: } x = -3, 2 & \quad f''(-3) = 7 > 0 = \text{minimum at } (-3, 4) \\
f''(2) = -6 < 0 & \quad f''(2) = -6 < 0 = \text{maximum at } (2, -5)
\end{align*}
\]