Section 3.8

1. Find y'' for $y = \sqrt{x^2+1}$.

$$y = (x^2+1)^{\frac{1}{2}}$$

$$y' = \underbrace{\tfrac{1}{2}(x^2+1)^{-\frac{1}{2}}}_{g}\underbrace{(2x)}_{h}$$

$$y'' = g'h + gh'$$

$$= \underbrace{-\tfrac{1}{4}(x^2+1)^{-\frac{3}{2}}(2x)}_{g'}\underbrace{(2x)}_{h} + \underbrace{\tfrac{1}{2}(x^2+1)^{-\frac{1}{2}}}_{g}\underbrace{(2)}_{h'}$$

2. If $r(t) = \langle t^3, t^2 \rangle$ represents the position of a particle at time t, find the angle between the velocity and the acceleration vector at time $t = 1$.

$$v(t) = r'(t) = \langle 3t^2, 2t \rangle \Rightarrow v(1) = \langle 3, 2 \rangle$$

$$a(t) = v'(t) = \langle 6t, 2 \rangle \Rightarrow a(1) = \langle 6, 2 \rangle$$

$$\cos\theta = \frac{\langle 3,2\rangle \cdot \langle 6,2\rangle}{|\langle 3,2\rangle||\langle 6,2\rangle|} = \frac{18+4}{\sqrt{13}\sqrt{40}}$$

inverse cosine

$$\boxed{\theta \approx 15^\circ}$$

3. Find the 98th derivative of:

a.) $f(x) = \frac{1}{x^2}$

$f = x^{-2}$

$f' = -2x^{-3}$

$f'' = 3 \cdot 2x^{-4}$

$f''' = -4 \cdot 3 \cdot 2x^{-5}$

$f^{4} = 5 \cdot 4 \cdot 3 \cdot 2x^{-6}$

$f^{n} = (-1)^{n}(n+1)!\, x^{-(n+2)}$

$f^{98} = (-1)^{98}\, 99!\, x^{-100}$

$$\boxed{f^{98} = \frac{99!}{x^{100}}}$$

b.) $f(x) = \sin(3x)$

$f' = \cos(3x) \cdot 3$

$f'' = -\sin(3x) \cdot 3^2$

$f''' = -\cos(3x) \cdot 3^3$

$f^{4} = \sin(3x) \cdot 3^4$

98 is not divisible by 4.

but 96 is.

$f^{96} = \sin(3x) \cdot 3^{96}$

$f^{97} = \cos(3x) \cdot 3^{97}$

$$\boxed{f^{98} = -\sin(3x) \cdot 3^{98}}$$

The slope of the parametric curve $x = f(t)$, $y = g(t)$ at $t = a$ is

$$m = \left.\frac{dy/dt}{dx/dt}\right|_{t=a}$$

Section 3.9

4. Given $x = \cos t$ and $y = t^2$, find the equation of the tangent line at $t = \frac{\pi}{4}$.

point $= \left(\frac{\sqrt{2}}{2}, \frac{\pi^2}{16}\right)$

$t = \frac{\pi}{4}$

$$m = \left.\frac{dy/dt}{dx/dt}\right|_{t=\frac{\pi}{4}} = \left.\frac{2t}{-\sin t}\right|_{t=\frac{\pi}{4}}$$

$$= \frac{2\left(\frac{\pi}{4}\right)}{-\frac{\sqrt{2}}{2}}$$

$$= \left(\frac{\pi}{2}\right)\left(-\frac{2}{\sqrt{2}}\right)$$

$$\boxed{m = -\frac{\pi}{\sqrt{2}}}$$

equation:

$$\boxed{y - \frac{\pi^2}{16} = \frac{-\pi}{\sqrt{2}}\left(x - \frac{\sqrt{2}}{2}\right)}$$

5. Let $x = t^4 - 4t^3$ and $y = 3t^2 - 6t$.

a.) Find the equation of the tangent line at the point $(5, 9)$.

find t-value that yields $(5,9)$

$x = 5 \Rightarrow \boxed{t^4 - 4t^3 = 5}$

$y = 9 \Rightarrow \frac{3t^2 - 6t}{3} = \frac{9}{3}$

$t^2 - 2t = 3$

$t^2 - 2t - 3 = 0$

$(t-3)(t+1) = 0$

$t = 3$

$\boxed{t = -1}$

which will yield $x = 5$?

$(-1)^4 - 4(-1)^3 = 5$

$$t = -1 \Rightarrow m = \left.\frac{dy/dt}{dx/dt}\right|_{t=-1}$$

$$= \left.\frac{6t-6}{4t^3 - 12t^2}\right|_{t=-1}$$

$$= \frac{-12}{-4-12} = \frac{12}{16} = \frac{3}{4}$$

point $= (5, 9)$

$$\boxed{\text{equation} \quad y - 9 = \frac{3}{4}(x-5)}$$

b.) Find all point(s) on the curve where the tangent line is vertical or horizontal.

$x = t^4 - 4t^3$

$y = 3t^2 - 6t$

vertical / horizontal tangents

① Find & simplify $\frac{dy/dt}{dx/dt} = \frac{6t-6}{4t^3 - 12t^2}$

$$= \frac{6(t-1)}{4t^2(t-3)}$$

$$= \frac{3(t-1)}{2t^2(t-3)}$$

vertical tangents: set simplified denominator $= 0$.

$t = 0 \Rightarrow$ point $= (0, 0)$

$t = 3 \Rightarrow$ point $= (-27, 9)$

horizontal tangents: set simplified numerator $= 0$

$t = 1 \Rightarrow$ point $(-3, -3)$

6. Show the curve $x = \cos t$ and $y = \sin t \cos t$ has two tangents at $\boxed{(0,0)}$. Find the equations of these tangent lines.

two tangents $\Rightarrow$ curve crosses itself (0,0)
$\Rightarrow$ two different parameters yield (0,0)

$x = \cos t = 0$
$y = \sin t \cos t = 0$

$\boxed{t = \frac{\pi}{2}, \frac{3\pi}{2}}$

① $t = \frac{\pi}{2}$
point $(0,0)$, $\quad m = \left.\dfrac{dy/dt}{dx/dt}\right|_{t=\frac{\pi}{2}} = \left.\dfrac{(\cos t)(\cos t) + (\sin t)(-\sin t)}{-\sin t}\right|_{t=\frac{\pi}{2}}$

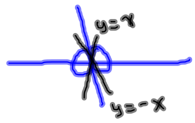
$= \left.\dfrac{\cos^2 t - \sin^2 t}{-\sin t}\right|_{t=\frac{\pi}{2}}$

$= \dfrac{-1}{-1}$

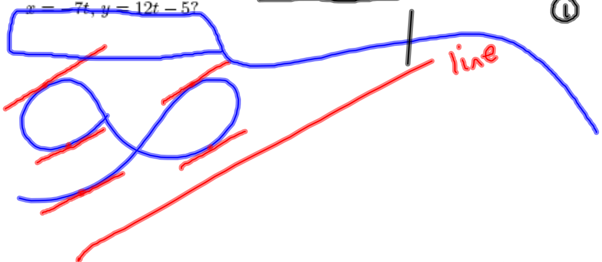
$\boxed{m = 1} \Rightarrow \boxed{y = x}$

② $t = \frac{3\pi}{2}$: $m = \left.\dfrac{\cos^2 t - \sin^2 t}{-\sin t}\right|_{t=\frac{3\pi}{2}}$

$m = \dfrac{-1}{1} = -1 \Rightarrow \boxed{y = -x}$



7. At what points on the curve $x = t^3 + 4t$, $y = 6t^2$ is the tangent line parallel to the line with equations $x = -7t$, $y = 12t - 5$?



① $m_1 = \dfrac{dy/dt}{dx/dt} = \dfrac{12t}{3t^2+4}$

$m_2 = \dfrac{dy/dt}{dx/dt} = \dfrac{12}{-7}$

parallel $\Rightarrow$ same slope

$\frac{1}{12} \cdot \dfrac{12t}{3t^2+4} = \dfrac{12}{-7} \cdot \frac{1}{12}$

$\dfrac{t}{3t^2+4} = -\dfrac{1}{7} \Rightarrow 7t = -(3t^2+4)$

$7t = -3t^2 - 4$

$x = t^3 + 4t$
$y = 6t^2$

$3t^2 + 7t + 4 = 0$

$(3t + 4)(t + 1) = 0$

$\boxed{t = -\frac{4}{3},\ t = -1}$

points?

$t = -1 \Rightarrow (-5, 6)$

$t = -\frac{4}{3} \Rightarrow \left(-\frac{208}{27}, 6\left(\frac{16}{9}\right)\right)$

8. Water leaking onto a floor creates a circular pool with an area that increases at a rate of 3 square inches per minute. How fast is the radius of the pool increasing when the radius is 10 inches?

let $A =$ area of circle

$r =$ radius of circle

Given: $\frac{dA}{dt} = 3$

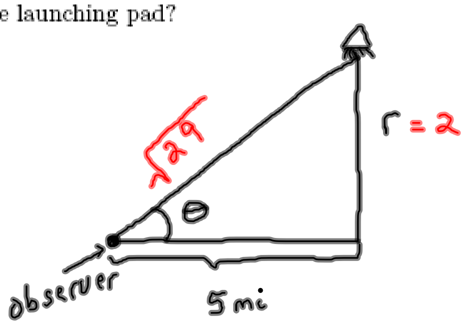
Find: $\left.\frac{dr}{dt}\right|_{r=10}$

$A = \pi r^2$

$\underbrace{\frac{dA}{dt}}_{3} = 2\pi \underbrace{r}_{10} \frac{dr}{dt} \Rightarrow 3 = 20\pi \frac{dr}{dt}$

$$\boxed{\frac{3}{20\pi}\,\frac{\text{in}}{\text{min}} = \frac{dr}{dt}}$$

9. When a rocket is 2 miles high, it is moving vertically upward at a speed of 300 mph. At that instant, how fast is the angle of elevation of the rocket increasing, as seen by an observer on the ground 5 miles from the launching pad?



Given: $\frac{dr}{dt} = \boxed{300\text{mph}}$ | $r = 2$ mi

Find $\frac{d\theta}{dt}$ at this time.

$$\cos\theta = \frac{5}{\sqrt{29}}$$

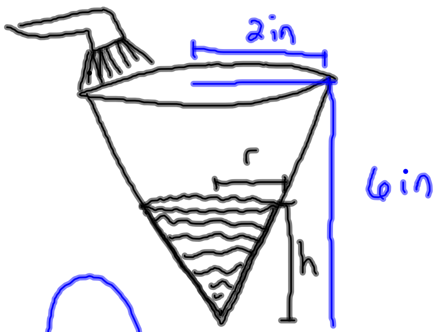
$$\tan\theta = \frac{r}{5}$$

$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{5}\cancelto{300}{\frac{dr}{dt}}$$

$$\frac{d\theta}{dt} = \cos^2\theta\,(60)$$

$$= \frac{25}{29}(60)\ \frac{\text{radians}}{\text{hour}}$$

10. A filter in the shape of a cone is 6 inches high and has a radius of 2 inches at the top. A solution is poured in the cone so that the water level is rising at a rate of $\frac{3}{2}$ inches per second. How fast is the water being poured in when the water level has a depth of 2 inches?.



h= water level

Given: $\frac{dh}{dt} = \frac{3}{2} \frac{in}{sec}$

Find $\left.\frac{dV}{dt}\right|_{h=2in}$

$V = \frac{1}{3}\pi r^2 h$

eliminate r from the equation:

similar cones,

$\frac{2}{6} = \frac{r}{h}$

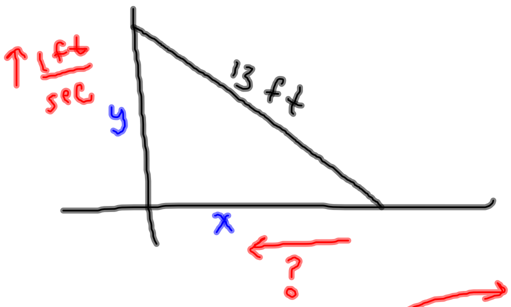
$\boxed{r = \frac{1}{3}h}$

$V = \frac{\pi}{27} h^3$

$\frac{dV}{dt} = \frac{\pi}{27} 3 \cancelto{4}{h^2} \cancelto{\frac{3}{2}}{\frac{dh}{dt}}$

$= \boxed{\frac{\pi}{27}(12)\left(\frac{3}{2}\right) \frac{in^3}{sec}}$

11. One end of a 13 foot ladder is on the ground, and the other end rests on a vertical wall. If the top of the ladder is being pushed up the wall at a rate of 1 foot per second, how fast is the base of the ladder approaching the wall when it is 3 feet from the wall?



given: $\frac{dy}{dt} = 1$

Find $\left.\frac{dx}{dt}\right|_{x=3}$

$$x^2 + y^2 = 13^2$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$x \to 3$, $y \to \sqrt{160}$, $\frac{dy}{dt} \to 1$

$x = 3$

$9 + y^2 = 13^2$

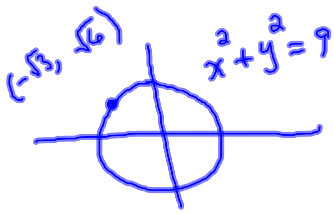
$y^2 = 160$

$y = \sqrt{160}$

$$\cancel{2}(3)\frac{dx}{dt} = -\cancel{2}\sqrt{160}\,(1)$$

$$\boxed{\frac{dx}{dt} = -\frac{\sqrt{160}}{3} \quad \frac{f}{s}}$$

12. A point moves around the circle $x^2 + y^2 = 9$. When the point is at $(-\sqrt{3}, \sqrt{6})$, its x coordinate is increasing at a rate of 20 units per second. How fast is its y coordinate changing at that instant?



Given: $\frac{dx}{dt} = 20 \Big|_{\substack{x=-\sqrt{3} \\ y=\sqrt{6}}}$

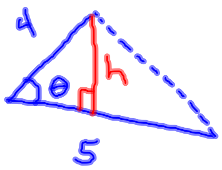
Find $\frac{dy}{dt}$ at this same instant.

$$x^2 + y^2 = 9 \implies 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$2(-\sqrt{3})(20) + 2\sqrt{6}\frac{dy}{dt} = 0$$

$$\cancel{2}\sqrt{6}\frac{dy}{dt} = \cancel{2}\sqrt{3}(20) \implies \boxed{\frac{dy}{dt} = \frac{20\sqrt{3}}{\sqrt{6}} \ \frac{\text{units}}{\text{second}}}$$

13. Two sides of a triangle have lengths 5 m and 4 m. The angle between them is increasing at a rate of 0.06 rad/s. Find the rate at which the area of the triangle is increasing when the angle between the sides of fixed length is 60°.



Given: $\frac{d\theta}{dt} = 0.06$

Find $\left.\frac{dA}{dt}\right|_{\theta = 60^\circ = \frac{\pi}{3}}$

$b = 5$

$\sin\theta = \frac{h}{4}$

$h = 4\sin\theta$

$A = \frac{1}{2}bh$

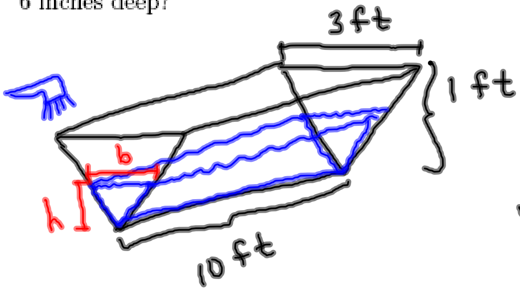
$A = \frac{1}{2}(5)(4\sin\theta) \Rightarrow A = 10\sin\theta$

$$\frac{dA}{dt} = 10\cos\theta\frac{d\theta}{dt}$$

$$= 10\left(\cos\frac{\pi}{3}\right)(0.06)$$

$$= \boxed{10\left(\frac{1}{2}\right)(0.06)\ \frac{m^2}{sec}}$$

14. A trough is 10 feet long and its ends have the shape of isosceles triangles that are 3 feet across the top and have a height of 1 foot. If the trough is filled with water at a rate of 12 cubic feet per minute, how fast is the water level rising when the water is 6 inches deep?



$$\frac{3}{1} = \frac{b}{h} \Rightarrow \boxed{b = 3h}$$

Given: $\frac{dV}{dt} = 12.$

Find $\left.\frac{dh}{dt}\right|_{h = 6\text{in} = \frac{1}{2}\text{ft}}$

h= height of water

b= base of water

keep r & h

$$V_{\text{water}} = \left(\tfrac{1}{2}bh\right)(10) = 5bh$$

$$= 5(3h)h$$

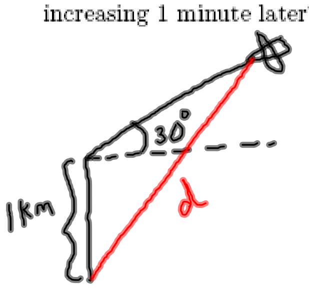
$$V = 15h^2$$

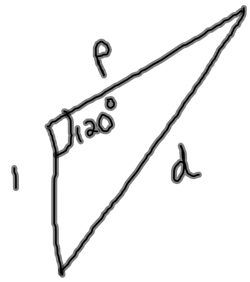
$$\frac{dV}{dt} = 30h\,\frac{dh}{dt}$$

$$12 = 30\left(\tfrac{1}{2}\right)\frac{dh}{dt}$$

$$\boxed{\frac{12}{15}\,\frac{f}{m} = \frac{dh}{dt}}$$

15. A plane flying with a constant speed of 300 km per hour passes over a radar station at an altitude of 1 km and climbs at an angle of 30°. At what rate is the distance from the plane to the radar station increasing 1 minute later?





Given: $\frac{dp}{dt} = 300 \frac{km}{hr}$

Find $\frac{dd}{dt}\Big|_{t = 1\text{ minute}}$

$t = \frac{1}{60}$ hour

$p = \frac{300}{60} = 5$ Km

$$d^2 = 1^2 + p^2 - 2(1)(p)\cos(120°) \quad -\tfrac{1}{2}$$

$$d^2 = 1 + p^2 - 2p\left(-\tfrac{1}{2}\right)$$

$\boxed{p = 5}$

$\frac{dp}{dt} = 300$

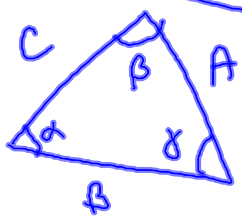
$d = \sqrt{31}$

$$d^2 = 1 + p^2 + p$$

$$2d\frac{dd}{dt} = 2p\frac{dp}{dt} + \frac{dp}{dt} = (2p+1)\frac{dp}{dt}$$

$$2\sqrt{31}\frac{dd}{dt} = (11)300 \Rightarrow \frac{dd}{dt} = \frac{300(11)}{2\sqrt{31}} \frac{km}{hr}$$

law of cosines



$$C^2 = A^2 + B^2 - 2AB\cos\gamma$$