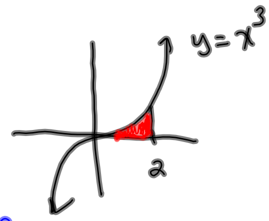
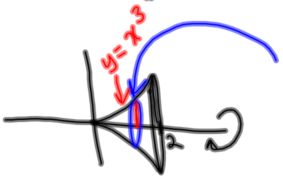
1. First, let's recap the disk and washer method:

a.) Find the integral that gives the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 0$, $x = 0$ and $x = 2$ about the x axis. Do not integrate.

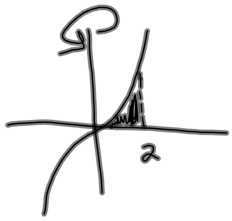


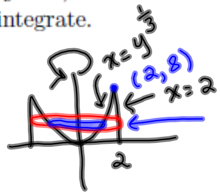


disk

$$V = \int_0^2 \pi (x^3)^2 \, dx$$

b.) Find the integral that gives the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 0$, $x = 0$ and $x = 2$ about the y axis. Do not integrate.

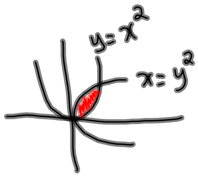


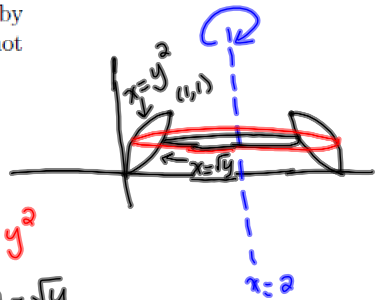


washer

$$V = \int_0^8 \pi \left[(2)^2 - \left(y^{\frac{1}{3}}\right)^2 \right] dy$$

c.) Find the integral that gives the volume of the solid obtained by rotating the region bounded by $y = x^2$ and $x = y^2$ about the line $x = 2$. Do not integrate.



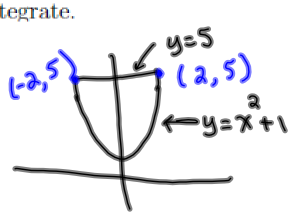


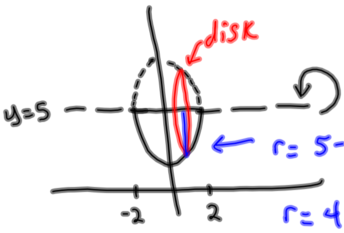
$R = 2 - y^2$

$r = 2 - \sqrt{y}$

$$V = \int_0^1 \pi \left[(2-y^2)^2 - (2-\sqrt{y})^2 \right] dy$$

d.) Find the integral that gives the volume of the solid obtained by rotating the region bounded by $y = x^2 + 1$, $y = 5$ about the line $y = 5$. Do not integrate.





$r = 5 - (x^2+1)$

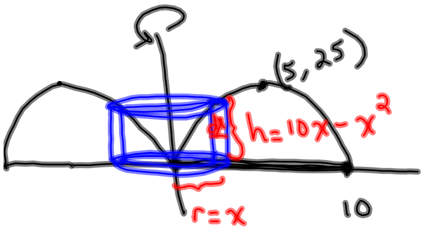
$r = 4 - x^2$

$$V = \int_{-2}^2 \pi \left[(4-x^2)^2 \right] dx$$

2. Find the volume of the solid obtained by rotating the region bounded by the given curve(s) about the specified axis.

$V_{shells} = (2\pi r)(h)\Delta r$

a.) $y = 10x - x^2$, $y = 0$ about the y axis.



vertex of parabola occurs when $y' = 0$

$10 - 2x = 0$

$x = 5$

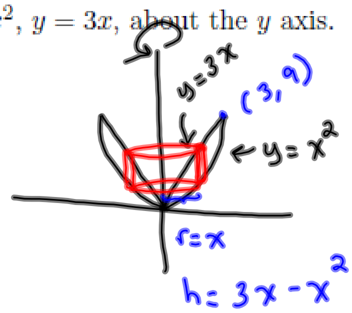
$y = 25$

$V = \int 2\pi r h \, dx$

$= \int_0^{10} 2\pi x (10x - x^2) \, dx$

$= 2\pi \int_0^{10} (10x^2 - x^3) \, dx$

b.) $y = x^2$, $y = 3x$, about the y axis.

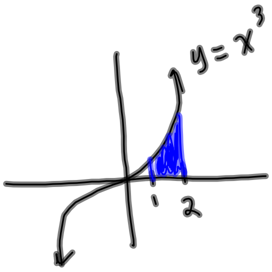


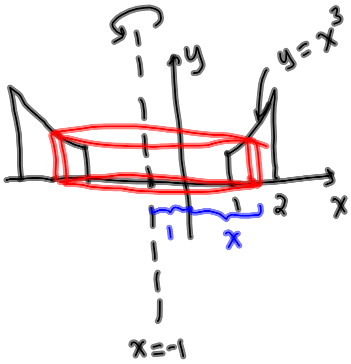
$V = \int_0^3 2\pi r h \, dx$

$= \int_0^3 2\pi x (3x - x^2) \, dx$

$= 2\pi \int_0^3 (3x^2 - x^3) \, dx$

c.) $y = x^3$, $y = 0$, $x = 1$, $x = 2$, about the line $x = -1$.





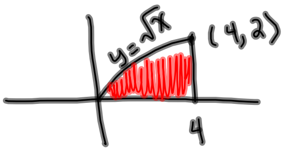
$r = x+1$

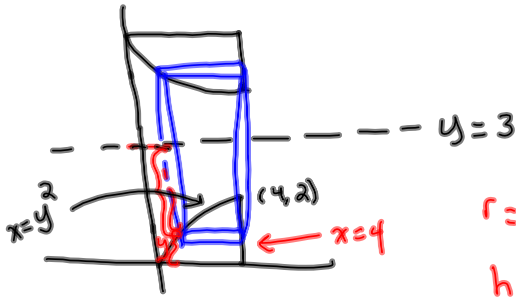
$h = x^3$

$$V = \int_1^2 2\pi(x+1)x^3\,dx$$

$$= 2\pi \int_1^2 \left(x^4 + x^3\right) dx$$

d.) $y = \sqrt{x}$, $x = 0$, $x = 4$, $y = 0$, about the line $y = 3$.





$r = 3-y$

$h = 4-y^2$

$$V = \int_0^2 2\pi(3-y)(4-y^2)\,dy$$

Foil & integrate

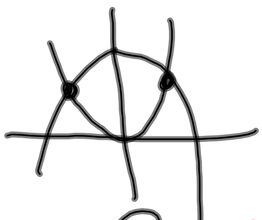
e.) $y = x^2$ and $y = 4 - x^2$, about the line $x = \sqrt{2}$.

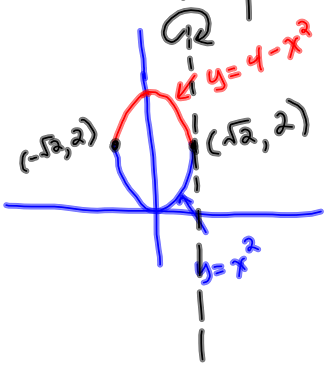
$x^2 = 4-x^2$

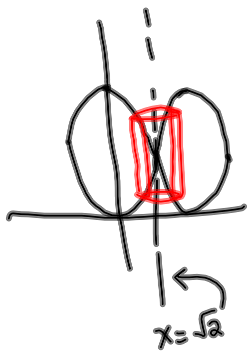
$2x^2 = 4$

$x^2 = 2$

$x = \pm\sqrt{2}$







$$\boxed{V = \int_{-\sqrt{2}}^{\sqrt{2}} 2\pi(\sqrt{2}-x)(4-2x^2)\,dx}$$

$r = \sqrt{2}-x$

$h = 4-x^2-(x^2) = 4-2x^2$

Section 7.4

3. How much work is done in lifting a 30 lb barbell from the floor to a height of 4 feet?

If force is constant, then $W = F \cdot d$

d = distance moved [m or feet]

F = force [N or pounds]

mass x acceleration

$W = (30 \text{ lbs})(4 \text{ feet})$

$\boxed{W = 120 \text{ ft lbs}}$

4. When a particle is at a distance x meters from the origin, a force of $f(x) = 3x^2 + 2$ Newtons acts on it. How much work is done in moving the object from $x = 2$ to $x = 4$?

object moving along a straight path from $x=a$ to $x=b$ via force $f(x)$, is

$$\boxed{W = \int_a^b f(x)\,dx}$$

$$W = \int_2^4 f(x)\,dx$$

$$= \int_2^4 (3x^2+2)\,dx$$

$$= \sim\sim \text{ N-m}$$

(Joule)

5. A spring has a natural length of 6 inches. If a 5-lb force is required to maintain it to a length of 18 inches, how much work is required to stretch it from 1 foot to 3 feet?

Hooke's Law: The force required to hold a spring x units beyond the natural length is $\boxed{f(x) = kx}$

The work to stretch it is $W = \int kx\,dx$

limits of integration measure beyond natural length.

#5. natural length $= 6$ in $= \frac{1}{2}$ ft

to hold it to 18 inches [1 foot beyond natural length]

requires 5 lbs.

Hooke's Law: $f(x) = kx$ → $\boxed{5 = k}$

5 lbs, 1 foot → $\boxed{f(x) = 5x}$

Find work done in stretching it from 1 foot to 3 feet is:

subtract natural length of $\frac{1}{2}$ foot

$$W = \int_{\frac{1}{2}}^{\frac{5}{2}} 5x\,dx$$

$3 - \frac{1}{2}$, $1 - \frac{1}{2}$

$$W = \frac{5}{2}x^2 \Big|_{\frac{1}{2}}^{\frac{5}{2}} \text{ ft lbs}$$

$\int kx\,dx$

6. Suppose 2 J of work is needed to stretch a spring 1 meter beyond its natural length. How much work is done in stretching this spring 3.5 m beyond its natural length?

given: $2 = \int_0^1 kx\,dx$

$$2 = k\frac{x^2}{2}\Big|_0^1$$

$$2 = \frac{k}{2}$$

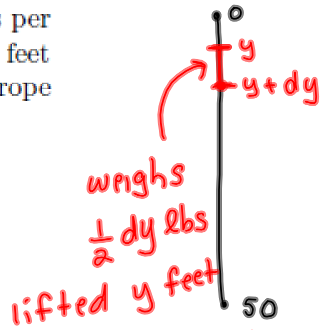
$$\boxed{4 = k}$$

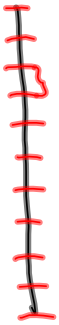
$f(x) = 4x$

$$W = \int_0^{3.5} 4x\,dx$$

$$\boxed{W = 2x^2\Big|_0^{3.5} \text{ J}}$$

7. A heavy rope, 50 feet long, weighs 0.5 pounds per foot and hangs over the edge of a building 120 feet high. How much work is done in pulling the rope to the top of the building?





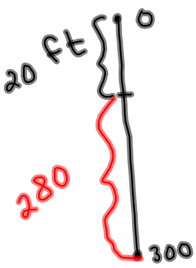
$$W = \int_0^{50} \frac{1}{2} y\,dy$$

$$W = \frac{1}{4} y^2 \Big|_0^{50} \text{ ft-lbs}$$

$$w = \left(\tfrac{1}{2}dy\right)(y)$$

In general, if the rope is ℓ feet long and weighs w $\frac{\text{lbs}}{\text{foot}}$ then work $= \int_0^{\ell} wy\,dy$

8. A 200 pound cable is 300 feet long and hangs vertically from the top of a tall building. How much work is required to pull 20 feet of the cable to the top of the building?



rope weighs $\frac{2}{3} \frac{\text{lb}}{\text{ft}}$

Pulling entire rope:

$$\int_0^{300} \frac{2}{3} y\,dy$$

pull only first 20 feet to the top:

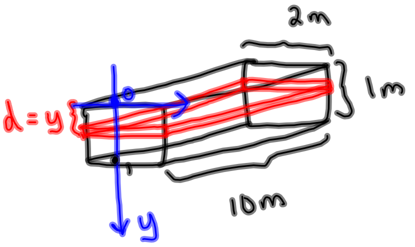
$$W = \int_0^{20} \frac{2}{3} y\,dy + \underbrace{(280)\left(\tfrac{2}{3}\right)}_{F}\underbrace{(20)}_{d}$$

$$W = \int_{20}^{300} \frac{2}{3} y\,dy$$

$$W = \frac{2}{3}\frac{y^2}{2}\Big|_0^{20} + (280)\left(\tfrac{2}{3}\right)(20) \text{ ft lbs}$$

$$= \frac{(20)^2}{3} + (280)\left(\tfrac{2}{3}\right)(20) \text{ ft lbs}$$

the water to the top.



let ρg = weight density of water.

$\rho g = 9800 \frac{N}{m^3}$

$W = F \cdot d$

F = weight of "slice" of water

$F = \rho g V$ $\quad V$ = volume of a slice of water.

$F = (\rho g)(20)\,dy$

$W = F \cdot d$, where $d = y$

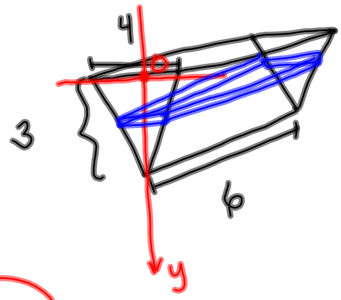
$W = 20 \rho g y \, dy$

To empty half the tank, $W = \int_0^{\frac{1}{2}} 20 \rho g y \, dy$

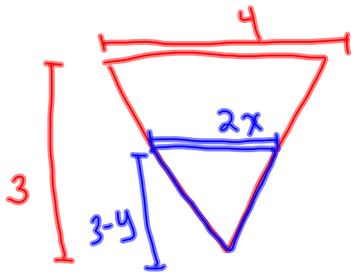
$W = 20 \rho g \frac{y^2}{2} \Big|_0^{\frac{1}{2}}$ J

$= \boxed{\frac{20(9800)\left(\frac{1}{2}\right)^2}{2} \text{ J}}$

10. A tank contains water and has the shape of a trough 6 feet long. The end of the trough is an isosceles triangle with height 3 feet and base length 4 feet. The vertex of the triangle is at the bottom. Find the work required to pump all of the water to the top of the tank.



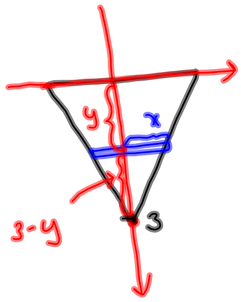
$F = \rho g V$



$V = (2x)(6)\,dy$

$\frac{4}{3} = \frac{2x}{3-y}$

$2x = \frac{4}{3}(3-y)$



$V = \frac{4}{3}(3-y)(6)\,dy = 8(3-y)\,dy$

here, since tank is in feet,

$\rho g = 62.5 \frac{lbs}{ft^3}$

$F = \rho g V = 8\rho g(3-y)\,dy$

$W = F \cdot d \qquad \boxed{d = y}$

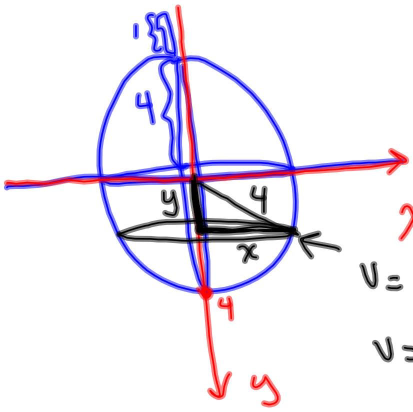
$W = 8\rho g\, y(3-y)\,dy$

Total work is $W = \int_0^3 8\rho g\, y(3-y)\,dy$

$W = 8(62.5)\int_0^3 (3y - y^2)\,dy$

11. A tank in the shape of sphere with radius 4 m is half full of water. The water is pumped from a spout at the top of the tank that is 1 m high. Find the work done in pumping the water through the spout.

half full!



$F = \rho g V$

$V = \pi r^2 \, dy$

$V = \pi x^2 \, dy$, where $x^2 + y^2 = 16$

$x^2 = 16 - y^2$

$V = \pi (16 - y^2) \, dy$

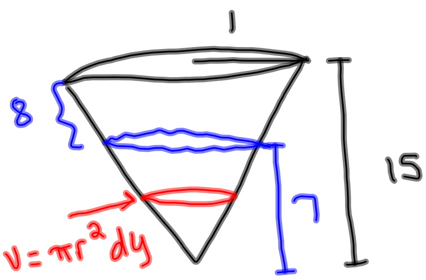
$F = \rho g \pi (16 - y^2) \, dy$

$W = F \cdot d$, $d = y + 5$

moves slice to top $\longrightarrow$ $W = \rho g \pi (16 - y^2)(y + 5) \, dy$

$\therefore$ Total work $= \int_0^4 \rho g \pi (16 - y^2)(y + 5) \, dy$

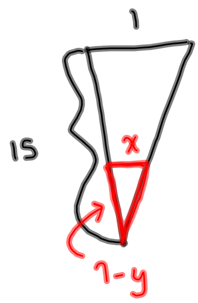
12. A tank in the shape of cone with radius 1 ~~inch~~ ft and height 15 ~~inches~~ ft is full of water to a depth of 7 ~~inches~~ ft. Find the work done in pumping the water to the top of the tank.

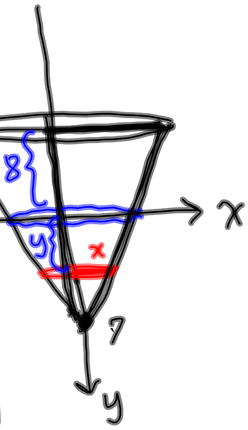


$V = \pi r^2 dy$

$V = \pi x^2 dy$

$F = \rho g V$





$\frac{1}{15} = \frac{x}{7-y}$

$x = \frac{1}{15}(7-y)$

$V = \pi\left(\frac{1}{15}(7-y)\right)^2 dy$

$F = \rho g V = \rho g \pi \left(\frac{1}{15}(7-y)\right)^2 dy$

$W = F \cdot d, \quad d = y+8$

$W = \rho g \pi \left(\frac{1}{15}(7-y)\right)^2 (y+8)\, dy$

$W = \int_0^7 \rho g \pi \left[\frac{1}{15}(7-y)\right]^2 (y+8)\, dy$