MATH 251 Spring 2017
EXAM II - VERSION A

LAST NAME: Key ________________ FIRST NAME: ________________

SECTION NUMBER: ____________

UIN: ________________________

DIRECTIONS:

1. You may use a calculator on this exam.
2. TURN OFF cell phones and put them away. If a cell phone is seen during the exam, your exam will be collected and you will receive a zero.
3. In Part 1 (Problems 1-10), mark the correct choice on your ScanTron using a No. 2 pencil. The ScanTron will not be returned, therefore *for your own records, also record your choices on your exam!* Each problem is worth 4 points.
4. In Part 2 (Problems 11-17), present your solutions in the space provided. *Show all your work* neatly and concisely and *clearly indicate your final answer.* You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.
5. Be sure to *write your name, section number and version letter of the exam on the ScanTron form.*

THE AGGIE CODE OF HONOR

"An Aggie does not lie, cheat or steal, or tolerate those who do."

Signature: ________________________

DO NOT WRITE BELOW!

Question Type	Points Awarded	Points
Multiple Choice		40
Free Response		60
Total		100

PART I: Multiple Choice. 4 points each.

$f(1,-2) = 1+4-\cancel{2}-8+\cancel{2}$

1. Which of the following statements is true regarding $f(x, y) = x^2 + y^2 - 2x + 4y + 2$?

   (a) $f(x, y)$ has a local maximum at $(1, -2, -3)$

   (b) $f(x, y)$ has a local minimum at $(1, -2, -3)$

   (c) $f(x, y)$ has a saddle point at $(1, -2, -3)$

   (d) $f(x, y)$ has no local extrema

   (e) The second derivative test for local extrema fails since $D < 0$

$f_x = 2x - 2 \quad f_x = 0 \quad x = 1$

$f_y = 2y + 4 \quad f_y = 0 \quad y = -2$

$f_{xx} = 2$

$f_{yy} = 2$

$f_{xy} = 0$

$D = f_{xx} f_{xy} - f_{xy}^2$

$D = 4 - 0 > 0$

$f_{xx} > 0$

local minimum at $(1, -2, -3)$

2. If $w = \ln(x + 2y^2 + 3)$, find $w_{xy}(3, 1)$.

   (a) $-\frac{1}{16}$

   (b) $\frac{1}{8}$

   (c) $-\frac{1}{64}$

   (d) $\frac{1}{4}$

   (e) $-\frac{1}{4}$

$w_x = \frac{1}{x+2y^2+3}$

$w_{xy} = \frac{-4y}{(x+2y^2+3)^2}$

$w_{xy}(3,1) = \frac{-4}{64}$

$= -\frac{1}{16}$

3. Find the absolute extrema of $f(x, y) = 1 + 4x - 5y$ on the set D, where D is the triangular region with vertices $(0, 0)$, $(0, 3)$ and $(2, 0)$.

   (a) The absolute maximum is $z = 32$, the absolute minimum is $z = -14$.

   (b) The absolute maximum is $z = 9$, the absolute minimum is $z = -14$.

   (c) The absolute maximum is $z = 1$, the absolute minimum is $z = 1$.

   (d) The absolute maximum is $z = 64$, the absolute minimum is $z = -14$.

   (e) The absolute maximum is $z = 32$, the absolute minimum is $z = 1$.

$f_x = 4$ no CP

$f_y = -5$

$y = -\frac{3}{2}x + 3$

Boundary $f(0,y) = -5y + 1, \quad 0 \le y \le 3$

$f(0,0) = 1, \quad f(0,3) = -14$

$f(x,0) = 4x + 1 \quad 0 \le x \le 2$

$f(0,0) = 1 \quad \boxed{f(2,0) = 9}$

$f(x, -\frac{3}{2}x+3) = 1 + 4x - (-\frac{3}{2}x + 3)$

$= 1 + 4x + \frac{3}{2}x - 3$

$= \frac{11x}{2} - 2, \quad 0 \le x \le 2$

$\boxed{f(0,3) = -14}$

$f(2,3) = -6$

4. Suppose we have a rectangle with length $l = 8$ cm and width $w = 6$ cm. Given there is a maximum error in measurement of 0.2 cm in the length, and a maximum error in measurement of 0.1 cm in the width, use differentials to estimate the maximum error in the calculated area of the rectangle.

(a) $dA = 2$ square centimeters

(b) $dA = 0.96$ square centimeters

(c) $dA = 2.2$ square centimeters

(d) $dA = 2.8$ square centimeters

(e) $dA = 1.4$ square centimeters

$A = lw$, $l = 8$, $dl = 0.2$

$w = 6$, $dw = 0.1$

$dA = l(dw) + w(dl)$

$= (8)(\frac{1}{10}) + (6)(\frac{1}{5})$

$= \frac{4}{5} + \frac{6}{5} = 2\,cm^2$

5. The level curves to the surface $f(x, y) = \dfrac{2x}{x^2 + y^2}$ are

(a) A family of hyperbolas

(b) A family of cones

(c) A family of cylinders

(d) A family of parabolas

(e) A family of circles

$k = \dfrac{2x}{x^2+y^2}$

$k(x^2+y^2) = 2x \rightarrow$ since k is constant, for example $k = 1$

circles

$x^2 + y^2 = 2x$ (complete square)

$x^2 - 2x + 1 + y^2 = 1$

$(x-1)^2 + y^2 = 1$ circle!

6. Suppose $u = (r^2 + 2s)^3$, $r = x + y \sin t$, $s = y + x \sin t$. Find $\dfrac{\partial u}{\partial x}$ when $x = 2$, $y = -3$, and $t = 0$.

(a) $\dfrac{\partial u}{\partial x} = 48$

(b) $\dfrac{\partial u}{\partial x} = -24$

(c) $\dfrac{\partial u}{\partial x} = -60$

(d) $\dfrac{\partial u}{\partial x} = -240$

(e) $\dfrac{\partial u}{\partial x} = 120$

$x = 2$, $r = 2 - 3\sin 0$ $\quad r = 2$

$y = -3$, $s = -3 + 2\sin 0$ $\quad s = -3$

$t = 0$

$\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial r}\dfrac{\partial r}{\partial x} + \dfrac{\partial u}{\partial s}\dfrac{\partial s}{\partial x}$

$= 3(r^2+2s)^2(2r)\cdot(1) + 2(r^2+2s)^2(2)(\sin t)$

$= 3(-2)^2(4) + 2(-2)^2 \cdot 2 \cdot 0$

$= 3(16)$

$= 48$

7. If $f(x, y) = y^2 \tan x$, find the directional derivative at the point $\left(\frac{\pi}{4}, -3\right)$ in the direction of $\theta = -\frac{\pi}{4}$

(a) $12\sqrt{2}$
(b) $15\sqrt{2}$
(c) $18\sqrt{2}$
(d) $\frac{9}{2}\sqrt{2} + 3\sqrt{2}$
(e) $\frac{9}{2}\sqrt{2} - 3\sqrt{2}$

$u = \langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \rangle$

$\nabla f = \langle y^2 \sec^2 x, 2y \tan x \rangle$

$\nabla f(\frac{\pi}{4}, -3) = \langle 9(2), -6 \rangle$

$= \langle 18, -6 \rangle$

$D_u f(\frac{\pi}{4}, -3) = \langle 18, -6 \rangle \cdot \langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \rangle$

$= 9\sqrt{2} + 3\sqrt{2}$

$= 12\sqrt{2}$

8. Find the equation of the tangent plane to the surface $z = e^{x^3+y^5}$ at the point $(1, -1, 1)$.

(a) $3x + 5y - z = -7$
(b) $3x - 5y - z = -9$
(c) $-3x - 5y - z = 3$
(d) $3x + 5y - z = -3$
(e) $-x - y + z = 1$

$z - 1 = f_x(1, -1)(x - 1) + f_y(1, -1)(y + 1)$

$f_x = 3x^2 e^{x^3+y^5}$

$f_y = 5y^4 e^{x^3+y^5}$

$z - 1 = 3(x - 1) + 5(y + 1)$

$z - 1 = 3x - 3 + 5y + 5$

$0 = 3x + 5y - z + 3$ or $3x + 5y - z = -3$

9. Find the directional derivative of $f(x, y) = x^2 - y^2 + 2xy$ at the point $P(2, -3)$ in the direction $\mathbf{i} - \mathbf{j}$.

(a) -12
(b) $\frac{8}{\sqrt{2}}$
(c) 10
(d) $-\frac{12}{\sqrt{2}}$
(e) $-\frac{10}{\sqrt{2}}$

$\vec{u} = \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$

$\nabla f = \langle 2x + 2y, -2y + 2x \rangle$

$\nabla f(2, -3) = \langle -2, 10 \rangle$

$D_u f(2, -3) = \langle -2, 10 \rangle \cdot \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$

$= -\frac{2}{\sqrt{2}} - \frac{10}{\sqrt{2}}$

$= \frac{-12}{\sqrt{2}}$

10. For any given surface $z = f(x, y)$, what is the geometric interpretation of $f_x(1, 2)$?

(a) The slope of the curve at the point $(1, 2, f(1, 2))$ obtained by intersecting the surface with the plane $y = 2$.

(b) The slope of the curve at the point $(1, 2, f(1, 2))$ obtained by intersecting the surface with the plane $x = 1$.

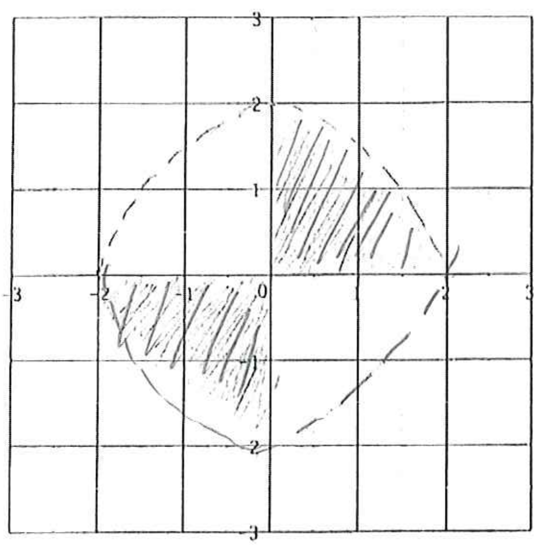
(c) The directional derivative of $f(x, y)$ at the point $(1, 2, f(1, 2))$ in the direction of $\mathbf{i}$.

(d) Both (a) and (c) are true.

(e) Both (b) and (c) are true.

Part II: Work out. Show all intermediate steps

11. (8 pts) Sketch the domain of $f(x, y) = \ln(4 - x^2 - y^2) + \sqrt{xy}$.



$4 - x^2 - y^2 > 0$

$xy \geq 0$ Q I, III

12. (6 pts) If If $f(x,y) = e^{xy} + ye^x$, find the maximum rate of change at the point $P(-1,1)$ and the direction in which it occurs. *Put your final answer in the blanks provided below*

$f_x = ye^{xy} + ye^x$

$f_y = xe^{xy} + e^x$

$\nabla f = \langle ye^{xy} + ye^x, \; xe^{xy} + e^x \rangle$

$= \langle e^{-1} + e^{-1}, \; -e^{-1} + e^{-1} \rangle$

$= \langle 2e^{-1}, 0 \rangle$

Maximum rate of change: $\sqrt{4e^{-2}} = 2e^{-1}$ Direction: $\langle 2e^{-1}, 0 \rangle$

13. (10 pts) Find all local extrema/saddle points for $f(x,y) = x^2 + y^2 + x^2y + 4$. *Put your final answer in the blanks provided below.* If there are none, write NONE in the answer space. If there is more than one, separate by a comma.

$f_x = 2x + 2xy$ $f_y = 2y + x^2$

$= 2x(1+y)$

$x = 0, y = -1$

$x = 0$: $f_y = 0$ $2y + x^2 = 0$ $y = 0$

$y = -1$: $-2 + x^2 = 0$ $x = \pm\sqrt{2}$

CP: $(0,0)$ $(\sqrt{2}, -1)$ $(-\sqrt{2}, -1)$

$f_{xx} = 2$, $f_{yy} = 2$

$f_{xy} = 2x$

CP	$D = 4 - 4x^2$	$f_{xx} = 2$	conclusion $f(x,y) = x^2 + y^2 + x^2y + 4$
$(0,0)$	$D = 4 > 0$	$f_{xx} > 0$	minimum at $(0,0,4)$
$(\sqrt{2}, -1)$	$D = -4 < 0$	$f_{xx} > 0$	saddle point $(\sqrt{2}, -1, 5)$
$(-\sqrt{2}, -1)$	$D = -4 < 0$	$f_{xx} > 0$	saddle point $(-\sqrt{2}, -1, 5)$

Max: none Min: $(0, 0, 4)$ SP: $(\pm\sqrt{2}, -1, 5)$

note: The formula $z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$ does not apply since $z \neq f(x,y)$

14. Consider the surface $z + e = xe^y \cos z$, find: $z - xe^y \cos z = -e$

a.) (8 pts) The equation of the tangent plane to the surface at the point $(1, 1, 0)$.

$F(x,y,z) = z - xe^y \cos z$

$\vec{n} = \nabla F = \langle -e^y \cos z, -xe^y \cos z, 1 + xe^y \sin z \rangle$

$\nabla F(1,1,0) = \langle -e, -e, 1 \rangle$

$\langle -e, -e, 1 \rangle \cdot \langle x-1, y-1, z \rangle = 0$

$-e(x-1) - e(y-1) + z = 0$

$-ex - ey + z + e + e = 0$ or $-ex - ey + z = -2e$

b.) (5 pts) The equation of the normal line to the surface at the point $(1, 1, 0)$.

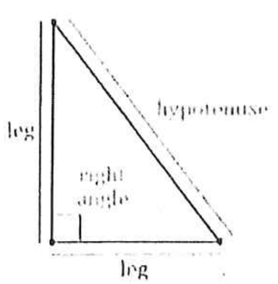
$r_0 + tv = \langle 1, 1, 0 \rangle + t \langle -e, -e, 1 \rangle$

$x = 1 - et$

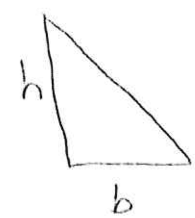
$y = 1 - et$

$z = t$

15. (7 pts) At a certain instant, the legs of a right triangle have lengths 2 and 4 feet, and they are increasing at the rates of 1 foot per minute and 2 feet per minute, respectively. How fast is the area of the triangle changing at that moment? Be sure to label all variables.



$A = \frac{1}{2} bh$



Given: when $b = 2$
$\frac{db}{dt} = 1$

when $h = 4$, $\frac{dh}{dt} = 2$

Find $\frac{dA}{dt} \Big/ \begin{smallmatrix} h=4 \\ b=2 \end{smallmatrix}$ $\frac{dh}{dt} = 2$, $\frac{db}{dt} = 1$

$A = A(bh)$

$\frac{dA}{dt} = \frac{1}{2} b \frac{dh}{dt} + \frac{1}{2} h \frac{db}{dt}$

$\frac{dA}{dt} = \frac{1}{2}(2)(2) + \frac{1}{2}(4)(1)$

$= 4 \text{ ft}^2/m$

16. (6 pts) Suppose we have 64 square feet of cardboard in order to make a closed rectangular box with length l, width w, and height h. Find a formula for the volume of the box in terms of l and w only that we would use in order to maximize the volume of this box. DO NOT SOLVE THE PROBLEM!!!! *Put your answer in the blank provided below.*

$A = 64$

$2lw + 2hw + 2lh = 64 \rightarrow h(2w + 2l) = 64 - lw$

$V = lwh$

$h = \dfrac{64 - 2lw}{2w + 2l}$

$h = \dfrac{32 - wl}{w + l}$

$V = lw\left(\dfrac{32 - wl}{w + l}\right)$

$V(l, w) =$ $\dfrac{32lw - w^2 l^2}{w + l}$, $l, w \geq 0$

17. (10 pts) Find the absolute extrema for $f(x, y) = 2x^3 + y^2 + 3$ on the set $D = \{(x, y) | x^2 + y^2 \leq 1\}$. Show all work. For the final answer, only the z coordinate is required, simplified. *Put your final answer in the blanks provided below.*

$f_x = 6x^2 \quad f_y = 2y \quad$ cn: $(0,0)$ so $\boxed{f(0,0) = 3}$

Boundary: $x^2 + y^2 = 1$, so $y = \pm\sqrt{1 - x^2}$

$f(x, \pm\sqrt{1 - x^2}) = 2x^3 + 1 - x^2 + 3$

$= 2x^3 - x^2 + 4, \quad -1 \leq x \leq 1$

$f'(x, \pm\sqrt{1 - x^2}) = 6x^2 - 2x, \quad -1 \leq x \leq 1$

$= 2x(3x - 1) \quad x = 0, \; x = \frac{1}{3}$

$x = 0$: $f(0, \pm 1) = 2 - 1 + 4 = 5$

$x = \frac{1}{3}$: $f(\frac{1}{3}, \pm\sqrt{8/9}) = \frac{107}{27}$

$\left.\begin{array}{l} f(1, 0) = 5 \\ f(-1, 0) = 1 \end{array}\right\}$ endpoints

Absoute Maximum : $z =$ 5 Absoute Minimum : $z =$ 1

18. THIS PROBLEM WAS ADDED FOR CONTENT. Use Lagrange Multipliers to find the absolute maximum value of $f(x, y, z) = xyz$ subject to the constraint $2x + y + 3z = 60$.

$\nabla f = \langle yz, xz, xy \rangle$

$\nabla g = \langle 2, 1, 3 \rangle$

$\nabla f = \lambda \nabla g$

$\langle yz, xz, xy \rangle = \lambda \langle 2, 1, 3 \rangle$

$yz = 2\lambda \qquad xyz = 2x\lambda$

$xz = \lambda \qquad xyz = y\lambda$

$xy = 3\lambda \qquad xyz = 3z\lambda$.

$2x\lambda = y\lambda \qquad 2x\lambda = 3z\lambda$

$\boxed{y = 2x} \qquad \boxed{z = \frac{2x}{3}}$

$2x + y + 3z = 60$

$2x + 2x + 2x = 60$

$6x = 60$

$x = 10$

$y = 20$

$z = \frac{20}{3}$

$f(10, 20, \frac{20}{3}) = \frac{4000}{3}$

19. THIS PROBLEM WAS ADDED FOR CONTENT. If $f(x, y) = 2xy - 3y - 2x + 5$, find the absolute maximum and minimum over the closed triangular region with vertices $(0, 0)$, $(2, 0)$ and $(2, 4)$. Put ALL consideration points in the table provided.

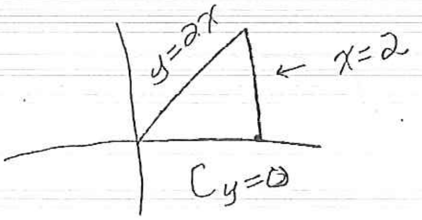
$f_x = 2y - 2 \qquad CP: \left(\frac{3}{2}, 1\right)$

$f_y = 2x - 3$

$f\left(\frac{3}{2}, 1\right) = 2\cdot\frac{3}{2} - 3 - 2\cdot\frac{3}{2} + 5$

$= 3 - 3 - 3 + 5$

$= 2$



critical points	function value
$\left(\frac{3}{2}, 1\right)$	2
$(0,0)$	5 min
$(2,0)$	1
$(4,2)$	5 max
$(1,2)$	1

$f(x, 0) = -3y + 5$

$f(0,0) = 5$

$f(2,0) = 1$

$f(y, 2) = 4y - 3y - 4 + 5$

$= y + 1$

$f(0,2) = 1$

$f(4,2) = 5$

$f(x, 2x) = 4x^2 - 6x - 2x + 5$

$= 4x^2 - 8x + 5$

$f'(x, 2x) = 8x - 8$

$CP: \; x = 1$
$y = 2$

$f(1,2) = 4 - 8 + 5$

$= 1$