

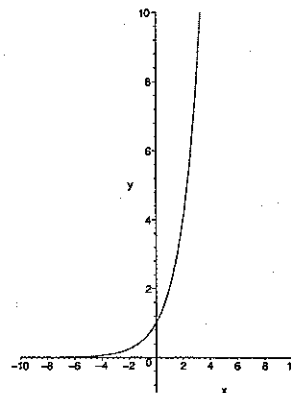
Section 4.1: Exponential Functions and their derivatives

Definition: The function $f(x) = a^x$, $a \neq 1$, $a > 0$, is called an exponential function.

- Case 1: Exponential Growth If $a > 1$, then $f(x) = a^x$ grows exponentially.

Note in particular:

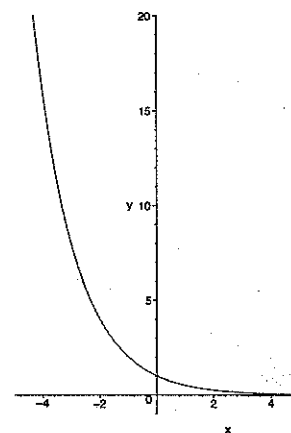
- The domain is $(-\infty, \infty)$.
- The range is $(0, \infty)$.
- $\lim_{x \rightarrow \infty} a^x = \infty$.
- $\lim_{x \rightarrow -\infty} a^x = 0$.



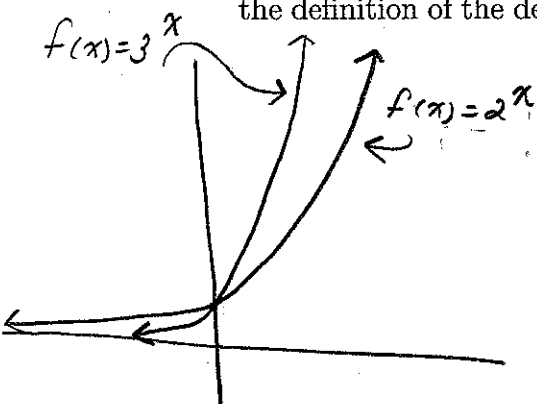
- Case 2: Exponential Decay If $0 < a < 1$, then $f(x) = a^x$ decays exponentially.

Note in particular:

- The domain is $(-\infty, \infty)$.
- The range is $(0, \infty)$.
- $\lim_{x \rightarrow \infty} a^x = 0$.
- $\lim_{x \rightarrow -\infty} a^x = \infty$.



EXAMPLE 1: Sketch the graph of $f(x) = 2^x$ and $g(x) = 3^x$ on the same axis. Using the definition of the derivative, estimate $f'(0)$ and $g'(0)$ by making a table of values.



$$f'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$$

$$f'(0) = \lim_{h \rightarrow 0} \frac{2^h - 1}{h}$$

$$g'(0) = \lim_{h \rightarrow 0} \frac{g(0+h) - g(0)}{h}$$

$$g'(0) = \lim_{h \rightarrow 0} \frac{3^h - 1}{h}$$

h	$f'(0) = \lim_{h \rightarrow 0} \frac{2^h - 1}{h}$	$g'(0) = \lim_{h \rightarrow 0} \frac{3^h - 1}{h}$
0.1	0.7177	1.1012
0.01	0.6956	1.1047
0.001	0.6934	1.0992
0.0001	0.6932	1.0987

$$f'(0) \approx \frac{d}{dx} 2^x \Big|_{x=0} \approx 0.69$$

$$g'(0) \approx \frac{d}{dx} 3^x \Big|_{x=0} \approx 1.1$$

there must be a number
between 2 and 3
so that $f'(0) = 1$

Definition: e is the number such that $\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$.

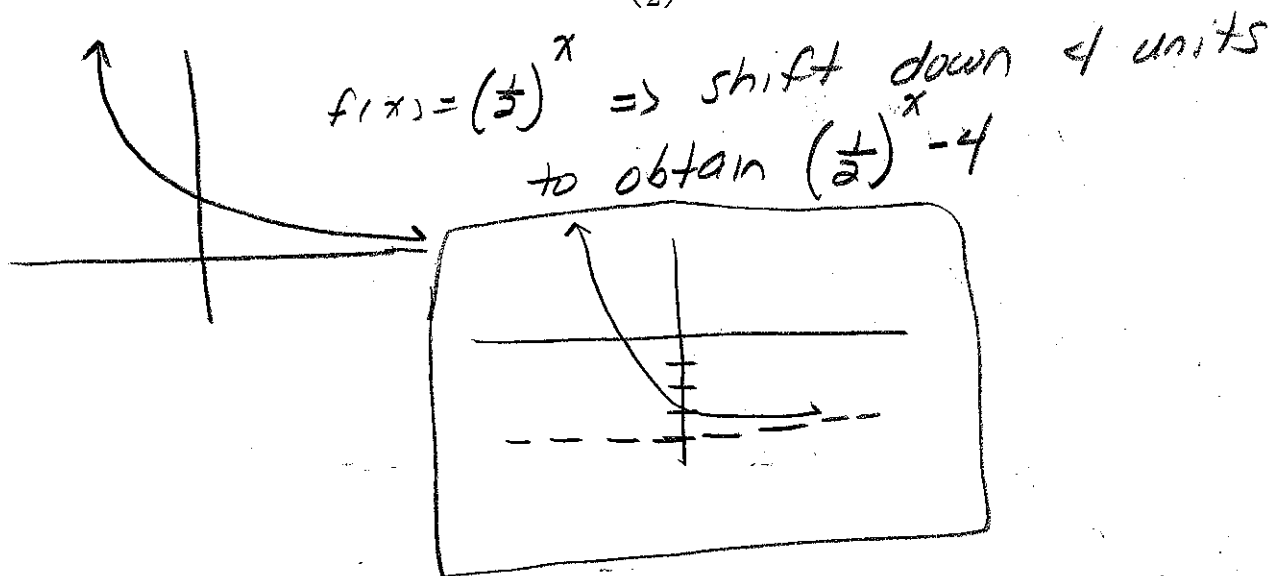
Recall If $f(x) = e^x$

$$f'(x) = \lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h}$$

$$= \lim_{h \rightarrow 0} \frac{e^x (e^h - 1)}{h}$$

$$= e^x$$

EXAMPLE 2: Sketch the graph of $f(x) = \left(\frac{1}{2}\right)^x - 4$ using transformations of graphs.



EXAMPLE 3: Find the limit:

(a) $\lim_{x \rightarrow -\infty} \left(\frac{\pi}{4}\right)^x$ since $0 < \frac{\pi}{4} < 1$, (exp decay)

$$\lim_{x \rightarrow -\infty} \left(\frac{\pi}{4}\right)^x = \infty$$

(b) $\lim_{x \rightarrow \infty} \left(\frac{2\pi}{7}\right)^x$

since $0 < \frac{2\pi}{7} < 1$ (exp decay)

$$\lim_{x \rightarrow \infty} \left(\frac{2\pi}{7}\right)^x = 0$$

$$(c) \lim_{x \rightarrow 2^+} \left(\frac{1}{4}\right)^{\frac{x}{2-x}}$$

$0 < \frac{1}{4} < 1 \Rightarrow$ exp decay

$$\lim_{x \rightarrow 2^+} \left(\frac{1}{4}\right)^{\frac{x}{2-x}} = \left(\frac{1}{4}\right)^{\lim_{x \rightarrow 2^+} \frac{x}{2-x}}$$

$$= \left(\frac{1}{4}\right)^{-\infty}$$

$$= \infty$$

$$(d) \lim_{x \rightarrow 1^-} e^{\frac{2}{x-1}} = e^{\lim_{x \rightarrow 1^-} \frac{2}{x-1}}$$

$$= e^{-\infty}$$

$$= 0$$

$$(e) \lim_{x \rightarrow 1^+} e^{\frac{2}{x-1}} = e^{\lim_{x \rightarrow 1^+} \frac{2}{x-1}}$$

$$= e^{\infty}$$

$$= \infty$$

$$(f) \lim_{x \rightarrow \infty} \frac{e^x - e^{-3x}}{e^{3x} + e^{-3x}} = \lim_{x \rightarrow \infty} \frac{e^x - \frac{1}{e^{3x}}}{e^{3x} + \frac{1}{e^{3x}}}$$

$$\lim_{x \rightarrow \infty} \frac{e^{-2x} - \frac{1}{e^{6x}}}{1 + \frac{1}{e^{6x}}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{e^{2x}} - \frac{1}{e^{6x}}}{1 + \frac{1}{e^{6x}}}$$

$$= 0$$

$$(g) \lim_{x \rightarrow -\infty} \frac{2^{-x} + 2^x}{4^{-x} + 3^x} \frac{4^x}{4^x}$$

$$\lim_{x \rightarrow -\infty} \frac{2^{-x} 4^x + 2^x 4^x}{1 + 3^x 4^x}$$

$$\lim_{x \rightarrow -\infty} \frac{2^x + 8^x}{1 + 12^x} = 0$$

$$\lim_{x \rightarrow -\infty} \frac{\frac{4^x}{2^x} + 8^x}{1 + 12^x}$$

Properties of Exponential Functions:

(i) $a^{x+y} = a^x a^y$

(ii) $a^{x-y} = \frac{a^x}{a^y}$

(iii) $(a^x)^y = a^{xy}$

(iv) $(ab)^x = a^x b^x$

Derivatives of Exponential Functions

(i) $\frac{d}{dx}e^x = e^x$

(ii) $\frac{d}{dx}e^{g(x)} = g'(x)e^{g(x)}$

EXAMPLE 4: Find the derivative.

(a) $y = \sqrt{e^x + x}$

$$y = (e^x + x)^{\frac{1}{2}}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2} (e^x + x)^{-\frac{1}{2}} (e^x + 1) \\ &= \frac{e^x + 1}{2\sqrt{e^x + x}} \end{aligned}$$

(b) $f(x) = e^{-5x} \cos(3x)$

$$\begin{aligned} f'(x) &= e^{-5x} (-3\sin(3x)) + -5e^{-5x} \cos(3x) \\ &= -e^{-5x} (3\sin 3x + 5\cos 3x) \end{aligned}$$

(c) $f(x) = e^{x \sin x}$

$$f'(x) = (x \cos x + \sin x) e^{x \sin x}$$

EXAMPLE 5: Find the equation of the tangent line to the graph of $2e^{xy} = x + y$ at the point ~~$(0, 2)$~~ $(0, 2)$

We will differentiate implicitly:

$$2e^{xy} = x + y$$

$$2(xy' + y)e^{xy} = 1 + y' \quad \text{plug in } x=0 \\ y=2$$

$$2(0 \cdot y' + 2)e^{0(2)} = 1 + y'$$

$$m = 3$$

$$4 = 1 + y'$$

$$y' = 3$$

$$y - 2 = 3(x - 0)$$

$$\boxed{y = 3x + 2}$$

EXAMPLE 6: For what value(s) of r does $y = e^{rx}$ satisfy $y + y' = y''$?

$$y = e^{rx}$$

$$y' = re^{rx}$$

$$y'' = r^2 e^{rx}$$

$$y + y' = y'' \Rightarrow e^{rx} + re^{rx} = r^2 e^{rx}$$

$$r^2 e^{rx} - re^{rx} - e^{rx} = 0$$

$$e^{rx} (r^2 - r - 1) = 0$$

$$r^2 - r - 1 = 0 \Rightarrow r = \frac{1 \pm \sqrt{1 - 4(-1)}}{2} = \boxed{\frac{1 \pm \sqrt{5}}{2}}$$

EXAMPLE 7: Find the equation of the tangent line to the parametric curve $x = e^{-t}$, $y = te^{2t}$ at $t = 0$.

$$\text{Recall: } m = \left. \frac{dy/dt}{dx/dt} \right|_{t=0}$$

$$m = \left. \frac{2te^{2t} + e^{2t}}{-e^{-t}} \right|_{t=0}$$

$$m = -1 \quad \text{point: } (1, 0)$$

$$y - 0 = -1(x - 1)$$

$$\boxed{y = -x + 1}$$

EXAMPLE 8: Find the derivative of $f(x) = g(e^x) + e^{g(\sin x)}$.

$$f'(x) = g'(e^x) \cdot e^x + g'(\sin x) \cos x e^{g(\sin x)}$$