Section 4.2: Inverse Functions

Definition: We say $f(x)$ is one-to-one provided whenever $f(x_1) = f(x_2)$, $x_1 = x_2$.

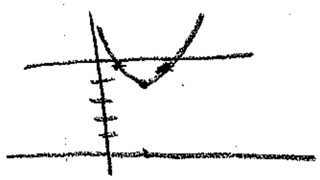
EXAMPLE 1: Prove $f(x) = x^2 - 2x + 5$ is not one-to-one.

we will graph it

vertex: $(1,4)$

$f(2) = 4 - 4 + 5 = 5$

$f(0) = 5$



$f(2) = f(0)$ but $2 \neq 0$

not one-to-one

Also fails horizontal line test

EXAMPLE 2: Prove $f(x) = 5 - 4x^3$ is one-to-one.

suppose $f(x_1) = f(x_2)$

$5 - 4x_1^3 = 5 - 4x_2^3$

$x_1^3 = x_2^3$

$x_1 = x_2$

EXAMPLE 3: Prove $f(x) = \dfrac{x-2}{x+2}$ is one-to-one.

$f(x_1) = f(x_2)$

$$\frac{x_1 - 2}{x_1 + 2} = \frac{x_2 - 2}{x_2 + 2}$$

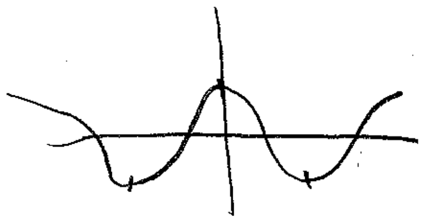
$(x_1 - 2)(x_2 + 2) = (x_2 - 2)(x_1 + 2)$

$x_1\cancel{x_2} + 2x_1 - 2x_2 - \cancel{4} = \cancel{x_1}x_2 + 2x_2 - 2x_1 - \cancel{4}$

$4x_1 = 4x_2$

$x_1 = x_2$

EXAMPLE 4: How can we restrict the domain of $f(x) = \cos x$ to make it one-to-one?

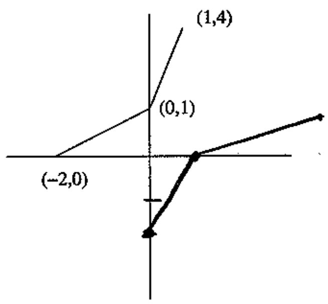


$f(x) = \cos x$
is one to one on $[0, \pi]$

Definition: Let $f(x)$ be a one-to-one function with domain D and range R. Then the inverse exists, denoted by $f^{-1}(x)$. Furthermore, the domain of f^{-1} = range of $f = R$ and the range of f^{-1} = domain of $f = D$. Moreover,

$$f(x) = y \iff f^{-1}(y) = x$$

EXAMPLE 5: Given the graph of f below, sketch the graph of f^{-1}.



EXAMPLE 6: Find the inverse and find the domain and range of the inverse.

(a) $f(x) = 5 - 4x^3$

domain $f = (-\infty, \infty) =$ range f^{-1}
range $f = (-\infty, \infty) =$ domain f^{-1}

$y = 5 - 4x^3$

$x = 5 - 4y^3 \Rightarrow 4y^3 = 5 - x$

$y^3 = \dfrac{5-x}{4}$

$y = \sqrt[3]{\dfrac{5-x}{4}}$

$\boxed{f^{-1}(x) = \sqrt[3]{\dfrac{5-x}{4}}, \text{ domain } f^{-1} = (-\infty, \infty), \text{ range } f^{-1} = (-\infty, \infty)}$

(b) $f(x) = \dfrac{2x+1}{1-3x}$ $\quad$ domain $f = \{x \mid x \neq \frac{1}{3}\} = \text{range } f^{-1}$

$$y = \frac{2x+1}{1-3x}$$

$$x = \frac{2y+1}{1-3y} \Rightarrow x(1-3y) = 2y+1$$

$$x - 3xy = 2y + 1$$

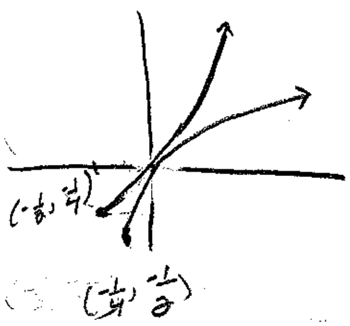
$$x - 1 = 2y + 3xy$$

$$x - 1 = y(2+3x)$$

$$y = \frac{x-1}{2+3x}$$

$$\boxed{\begin{array}{l} f^{-1} = \dfrac{x-1}{2+3x} \\ \text{dom } f^{-1} = \{x \mid x \neq -\frac{2}{3}\} \\ \text{range } f^{-1} = \{y \mid y \neq \frac{1}{3}\} \end{array}}$$

(c) $f(x) = x^2 + x$, for $x \geq -\dfrac{1}{2}$



domain $f = [-\frac{1}{2}, \infty) = \text{range } f^{-1}$

range $f = [-\frac{1}{4}, \infty) = \text{domain } f^{-1}$

$$y = x^2 + x$$

$$x = y^2 + y \Rightarrow y^2 + y - x = 0$$

$a = 1$

$b = 1$

$c = x$

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y = \frac{-1 \pm \sqrt{1+4x}}{2} \qquad y = \frac{-1 + \sqrt{1+4x}}{2}$$

Domain: $[-\frac{1}{4}, \infty)$

range: $[-\frac{1}{2}, \infty)$

Theorem: Suppose f is a one-to-one differentiable function with inverse function $g = f^{-1}$. Then g is differentiable and $g'(a) = \dfrac{1}{f'(g(a))}$.

EXAMPLE 7: Suppose g is the inverse of f and $f(2) = 3$, $f'(2) = 7$, $f(3) = 4$ and $f'(3) = \frac{1}{2}$. Find $g'(3)$.

since $f(2) = 3$

$f^{-1}(3) = 2,$

thus $g(3) = 2$

$$g'(3) = \frac{1}{f'(g(3))} = \frac{1}{f'(2)} = \boxed{\frac{1}{7}}$$

EXAMPLE 8: Suppose g is the inverse of f. Find $g'(4)$ if $f(x) = 3 + x + e^x$.

since $f(0) = 4$

$f^{-1}(4) = 0$

thus $g(4) = 0$

$$g'(4) = \frac{1}{f'(g(4))} = \frac{1}{f'(0)} = \boxed{\frac{1}{2}}$$

$f'(x) = 1 + e^x$

$f'(0) = 2$

EXAMPLE 9: Consider $f(x) = x^3$.

(a) Prove $f(x)$ is one-to-one.

$f(x_1) = f(x_2)$

$x_1^3 = x_2^3$

$x_1 = x_2$

(b) Use the above theorem to find $g'(8)$, where g is the inverse of f.

$f'(x) = 3x^2$

$$g'(8) = \frac{1}{f'(g(8))} = \frac{1}{f'(2)} = \boxed{\frac{1}{12}}$$

(c) Find $g(x)$

let $y = x^3$

$x = y^3$

$y = \sqrt[3]{x}$

(d) Find $g'(8)$ using part (c) and compare with your answer in part (b).

$g(x) = x^{\frac{1}{3}}$

$g'(x) = \frac{1}{3} x^{-\frac{2}{3}}$

$= \frac{1}{3x^{2/3}} \Rightarrow g'(8) = \frac{1}{3(8)^{2/3}}$

$= \frac{1}{3(4)} = \frac{1}{12}$

(e) Sketch the graph of f and g on the same axis.