Section 10.2: Series

Def: A *SERIES* a sum of a sequence, that is $\sum\limits_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + ... + a_{100} + ... + ...$ The goal is to determine when the series has a finite sum.

Def: If $\sum\limits_{n=1}^{\infty} a_n = S$, where S is finite, then we say the series *CONVERGES* and it's sum is S. If $\sum\limits_{n=1}^{\infty} a_n = \infty$ or does not exist, then we say the series *DIVERGES*.

EXAMPLE 1: Consider the following series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ... + \frac{1}{2^{100}} + ... + ...$$

The convergence (finite sum) or divergence (infinite sum) is not immediately obvious. To investigate this further, we will form the *SEQUENCE OF PARTIAL SUMS*.

Def: Let $\sum\limits_{n=1}^{\infty} a_n$ be a series. We will construct the sequence of partial sums $\{s_n\} = \{s_1, s_2, s_3, ..., ...\}$ as follows:

$s_1 = a_1$

$s_2 = a_1 + a_2$

$s_3 = a_1 + a_2 + a_3$

Therefore a general formula for s_n, the n^{th} term of the sequence, is

$s_n = \sum\limits_{i=1}^{n} a_i = a_1 + a_2 + ... + a_n$.

If $\lim\limits_{n\to\infty} s_n = S$, where S is finite, then we say the series $\sum\limits_{n=1}^{\infty} a_n$ converges and it's **sum** is S, that is $\sum\limits_{n=1}^{\infty} a_n = S$. If $\lim\limits_{n\to\infty} s_n$ is infinite or does not exist, then we say the series $\sum\limits_{n=1}^{\infty} a_n$ diverges.

Let's go back to the series in example 1:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ... + \frac{1}{2^{100}} + ... + ...$$

And form the sequence of partial sums.

$$s_1 = \sum_{i=1}^{1} \frac{1}{2^i} = \frac{1}{2} = 0.5$$

$$s_2 = \sum_{i=1}^{2} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{2^2} = 0.75$$

$$s_3 = \sum_{i=1}^{3} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} = 0.875$$

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$$s_{25} = \sum_{i=1}^{25} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ... + \frac{1}{2^{25}} \approx .999999702$$

It appears that

$$\lim_{n \to \infty} s_n = 1$$

Hence, we say $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges and it's sum is 1, that is,

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

EXAMPLE 2: Compare the partial sums for the series $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

$\sum_{n=1}^{\infty} \frac{1}{n}$	$\sum_{n=1}^{\infty} \frac{1}{n^2}$
$s_1 = \sum_{i=1}^{1} \frac{1}{i} = \frac{1}{1} = 1$	$s_1 = \sum_{i=1}^{1} \frac{1}{i^2} = \frac{1}{1} = 1$
$s_2 = \sum_{i=1}^{2} \frac{1}{i} = \frac{1}{1} + \frac{1}{2} = 1.5$	$s_2 = \sum_{i=1}^{2} \frac{1}{i^2} = \frac{1}{1} + \frac{1}{4} = 1.25$
$s_3 = \sum_{i=1}^{3} \frac{1}{i} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} \approx 1.83$	$s_3 = \sum_{i=1}^{3} \frac{1}{i^2} = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} \approx 1.36$
$s_4 = \sum_{i=1}^{4} \frac{1}{i} \approx 2.083$	$s_4 = \sum_{i=1}^{4} \frac{1}{i^2} \approx 1.424$
$s_5 = \sum_{i=1}^{5} \frac{1}{i} \approx 2.283$	$s_5 = \sum_{i=1}^{5} \frac{1}{i^2} \approx 1.464$
$s_6 = \sum_{i=1}^{6} \frac{1}{i} \approx 2.45$	$s_6 = \sum_{i=1}^{6} \frac{1}{i^2} \approx 1.49$
$s_7 = \sum_{i=1}^{7} \frac{1}{i} \approx 2.59$	$s_7 = \sum_{i=1}^{7} \frac{1}{i^2} \approx 1.51$
$s_8 = \sum_{i=1}^{8} \frac{1}{i} \approx 2.71$	$s_8 = \sum_{i=1}^{8} \frac{1}{i^2} \approx 1.52$
$s_9 = \sum_{i=1}^{9} \frac{1}{i} \approx 2.83$	$s_9 = \sum_{i=1}^{9} \frac{1}{i^2} \approx 1.54$
$s_{10} = \sum_{i=1}^{10} \frac{1}{i} \approx 2.93$	$s_{10} = \sum_{i=1}^{10} \frac{1}{i^2} \approx 1.55$
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$s_{20} = \sum_{i=1}^{20} \frac{1}{i} \approx 3.597$	$s_{20} = \sum_{i=1}^{20} \frac{1}{i^2} \approx 1.596$
$s_{30} = \sum_{i=1}^{30} \frac{1}{i} \approx 3.99$	$s_{30} = \sum_{i=1}^{30} \frac{1}{i^2} \approx 1.612$
$s_{40} = \sum_{i=1}^{40} \frac{1}{i} \approx 4.27$	$s_{40} = \sum_{i=1}^{40} \frac{1}{i^2} \approx 1.62$
$s_{50} = \sum_{i=1}^{50} \frac{1}{i} \approx 4.499$	$s_{50} = \sum_{i=1}^{50} \frac{1}{i^2} \approx 1.625$
$s_{100} = \sum_{i=1}^{100} \frac{1}{i} \approx 5.18$	$s_{100} = \sum_{i=1}^{100} \frac{1}{i^2} \approx 1.634$
$s_{1000} = \sum_{i=1}^{1000} \frac{1}{i} \approx 7.48$	$s_{1000} = \sum_{i=1}^{1000} \frac{1}{i^2} \approx 1.643$
$s_{10000} = \sum_{i=1}^{10000} \frac{1}{i} \approx 9.78$	$s_{10000} = \sum_{i=1}^{10000} \frac{1}{i^2} \approx 1.6448$
$s_{10^{10}} = \sum_{i=1}^{10^{10}} \frac{1}{i} \approx 23.60$	$s_{10^{10}} = \sum_{i=1}^{10^{10}} \frac{1}{i^2} \approx 1.6449$

It appears that $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ since the partial sums grow. However, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ appears to be approximately 1.6. Why do you think one diverged and one converged?

Test for Divergence: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges. NOTE: The converse is not necessarily true: If $\lim_{n\to\infty} a_n = 0$, then the series $\sum_{n=1}^{\infty} a_n$ does not necessarily converge. Therefore if you find that $\lim_{n\to\infty} a_n = 0$, then the divergence test fails and thus another test must be applied.

EXAMPLE 3: Use the Test For Divergence to determine whether the series converges or diverges:

a.) $\sum_{n=1}^{\infty} \frac{n^2}{3(n+1)(n+2)}$

b.) $\sum_{n=1}^{\infty} \ln\left(\frac{n}{5n+1}\right)$

c.) $\sum_{n=2}^{\infty} \frac{n}{\ln n}$

EXAMPLE 4: Explain why the test for divergence fails when applied to the series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

EXAMPLE 5: If the n^{th} partial sum of the series $\sum_{n=1}^{\infty} a_n$ is $s_n = \dfrac{n+1}{2n+4}$, find:

a.) The sum of the series.

b.) A general formula for a_n

Def: A *TELESCOPING SERIES* is a series of the form $\sum_{n=1}^{\infty}(a_{n+i} - a_n)$ for some integer $i \geq 1$. The series 'collapses'.

EXAMPLE 6: Determine whether the following telescoping series converges or diverges. If it converges, find the sum.

a.) $\sum_{n=1}^{\infty}\left(\sin\frac{1}{n} - \sin\frac{1}{n+1}\right)$

b.) $\sum_{n=1}^{\infty}\ln\frac{n+1}{n+2}$

c.) $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$

Def: The *GEOMETRIC SERIES* $\sum_{n=1}^{\infty} ar^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \geq 1$. If $|r| < 1$, then the sum is $\sum_{n=1}^{\infty} ar^{n-1} = \dfrac{a}{1-r}$.

EXAMPLE 7: Determine whether the following geometric series converges or diverges. If it converges, find the sum. If it diverges, explain why.

a.) $\sum_{n=1}^{\infty} 5\left(\frac{2}{7}\right)^n$

c.) $\sum_{n=0}^{\infty} \dfrac{2^{3n}}{(-5)^{n+1}}$

b.) $\sum_{n=0}^{\infty} \frac{(-1)^{n+1} + 3^n}{4^{n+1}}$

d.) $9 + 2 + \frac{4}{7} + \frac{8}{49} + ...$

EXAMPLE 8: Consider $\sum_{n=1}^{\infty}(x-5)^n$. Find the value(s) of x for which the series converges. Find the sum of the series for those values of x.