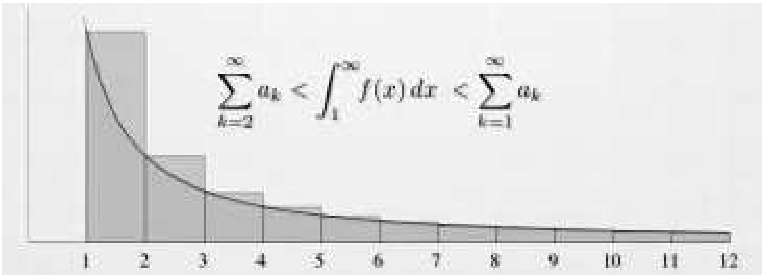
Section 10.3: The Integral and Comparison Tests

The Integral Test: If $f(x)$ is a positive, continuous, decreasing function on $[1, \infty]$, and $a_k = f(k)$. Then $\sum_{k=1}^{\infty} a_k$ and $\int_1^{\infty} f(x)\,dx$ either both converge or both diverge. Why is this so?



EXAMPLE 1: Determine whether the following series converge or diverge.

a.) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

b.) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

EXAMPLE 2: For what value(s) of p does the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge?

The p-series Test: A **p-series** is a series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + ...$, where $p > 0$. The special case where $p = 1$ is called the **harmonic series:**

$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + ...$ We showed earlier that this series diverges.

More specifically, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ **is convergent if** $p > 1$ **and divergent if** $p \leq 1$**.**

EXAMPLE 3: Determine whether the series $\sum_{n=1}^{\infty} \frac{1000}{n\sqrt{n}}$ converges or diverges.

Note that if $\sum a_n$ is *similar* to a p series, then we may want to *compare* the series to a p series.

The Comparison Test: Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{i=1}^{\infty} b_n$ are series of **positive terms**.

- If $\sum_{n=1}^{\infty} b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ is also convergent.
- If $\sum_{n=1}^{\infty} b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ is also divergent.

EXAMPLE 4: If $\sum a_n$ and $\sum b_n$ are series of **positive terms** and $\sum b_n$ is known to converge.

(a) If $a_n > b_n$ for all n what can be said about $\sum a_n$? Why?

(b) If $a_n < b_n$ for all n what can be said about $\sum a_n$? Why?

EXAMPLE 5: Determine whether the following series converge or diverge.

a.) $\sum_{n=1}^{\infty} \frac{n^4}{n^8 + n^2 + 1}$

b.) $\sum_{n=1}^{\infty} \frac{1 + 5^n}{4^n - n}$

c.) $\sum_{n=17}^{\infty} \frac{n}{\sqrt{n} - 4}$

d.) $\sum\limits_{n=1}^{\infty} \dfrac{\sin^2 n}{n\sqrt{n}}$

The Limit Comparison Test: If the Comparison Test is inconclusive, then we may apply the Limit Comparison Test:

Suppose $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ are series of positive terms.

If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = c > 0$, then either both series converge or both diverge.

Note: If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 0$ or $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \infty$, then the test fails and therefore we need to apply another test.

EXAMPLE 6: Determine whether the following series converge or diverge.

a.) $\sum_{n=1}^{\infty} \frac{n^2 - 5n}{n^3 + 2n}$

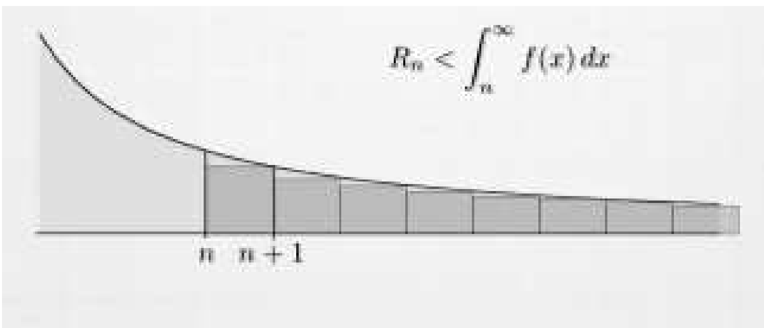
b.) $\sum_{n=1}^{\infty} \frac{1}{5^n - n}$

c.) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

Remainder Estimate: Suppose $s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + ... + a_n$ is the n^{th} partial sum of the convergent series $\sum_{n=1}^{\infty} a_n$. Then the **remainder** in using s_n to approximate the sum S is defined to be $R_n = S - s_n = \sum_{i=n+1}^{\infty} a_i = a_{n+1} + a_{n+2} + ...$

Moreover, if $\sum_{n=1}^{\infty} a_n$ was shown to be convergent by the integral test where $a_n = f(n)$, then

$$R_n = \sum_{i=n+1}^{\infty} a_i < \int_n^{\infty} f(x)\, dx.$$



EXAMPLE 7: Consider $\sum\limits_{n=1}^{\infty} \frac{1}{n^4}$

a.) Find the sum of the first 10 terms.

b.) Estimate the error

c.) Find the sum correct to 10 decimal places.

EXAMPLE 8: Consider $\sum_{n=1}^{\infty} \frac{3+\cos n}{n^5}$

a.) Prove the series converges.

b.) Approximate the sum of the series using s_6.

c.) By comparing the series to a p-series, estimate the error in using s_6 to approximate the sum of the series.

EXAMPLE 9: How many terms of the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ would you need to add to find its sum to within 0.01?