Section 10.4: Other Convergence Tests

Def: An alternating series is a series whose terms alternate signs. For example, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...$ is an alternating series. We would like to know under what conditions does an alternating series converge?

The Alternating Series Test: The alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$, where $a_n > 0$, converges if it satisfies both conditions given below:

- $a_{n+1} \leq a_n$ for all n (ie the sequence $\{a_n\}$ is decreasing).
- $\lim_{n\to\infty} a_n = 0$

Illustration as to why this is true.

Consider $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$, where $a_n > 0$ and a_n decreases to zero.

$s_1 = a_1$

$s_2 = a_1 - a_2$

$s_3 = a_1 - a_2 + a_3$

$s_4 = a_1 - a_2 + a_3 - a_4$

$s_5 = a_1 - a_2 + a_3 - a_4 + a_5$

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Plot the sequence of partial sums below:

EXAMPLE 1: Determine whether the following series are convergent.

a.) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+3}}$

b.) $\frac{1}{\ln 2} - \frac{1}{\ln 3} + \frac{1}{\ln 4} - \frac{1}{\ln 5} + ..$

c.) $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{1+n^2}$

Def: A series is **absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ converges. If $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ diverges, then the series is **conditionally convergent**. Note: If $\sum_{n=1}^{\infty} a_n$ is a series of positive terms, and $\sum_{n=1}^{\infty} a_n$ converges, then by default it is absolutely convergent.

EXAMPLE 2: Determine whether the following series are convergent, absolutely convergent, or divergent.

a.) $\sum_{n=1}^{\infty} \frac{1}{n^4}$

b.) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n}}$

c.) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$

d.) $\sum_{n=1}^{\infty} \frac{(-1)^n (n+1)}{\pi^4 n}$

Remainder Estimate and The Alternating Series Theorem

If $\sum_{n=1}^{\infty}(-1)^n a_n$, $a_n > 0$, is a convergent alternating series, and a partial sum $s_n = \sum_{i=1}^{n}(-1)^i a_i$ is used to approximate the sum of the series with remainder R_n, then

$$|R_n| \leq a_{n+1}$$

EXAMPLE 3: Consider $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}$

a.) Prove the series is absolutely convergent.

b.) Use s_6 to approximate the sum of the series and estimate the error (remainder).

EXAMPLE 4: How many terms of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!4^n}$ do we need to add together to approximate the sum to within $\frac{1}{100}$?

EXAMPLE 5: Approximate $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ with error less than 10^{-4}.

The Ratio Test:

- If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

- If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

- If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, then the test fails.

EXAMPLE 6: Determine whether the following series converge or diverge.

a.) $\sum_{n=1}^{\infty} \frac{(-2)^n}{n5^{n+1}}$

b.) $\sum_{n=1}^{\infty} \frac{(2n+1)!}{n!10^n}$

EXAMPLE 7: We showed the series $\sum_{n=1}^{\infty} \frac{(-2)^n}{n5^{n+1}}$ converges absolutely by the ratio test. Use s_2 to approximate the sum of the series and estimate the error.