Section 10.6: Representations of Functions as Power Series

Recall a geometric series is a series of the form $\sum_{n=1}^{\infty} ar^{n-1}$. If $|r| < 1$, $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$. Thus we can view $\sum_{n=0}^{\infty} x^n$ as a geometric series, and will converge provided $|x| < 1$. Moreover, if $|x| < 1$, then $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. In this section, we will start with the sum and write it as a power series with the associated radius of convergence.

EXAMPLE 1: Express the following functions as a power series. Include the radius of convergence.

a.) $f(x) = \dfrac{1}{1-3x}$

b.) $f(x) = \dfrac{1}{2x^4+1}$

c.) $f(x) = \dfrac{x}{2-x}$

d.) $f(x) = \dfrac{1}{16x^4 - 3}$

THEOREM If $f(x) = \sum_{n=0}^{\infty} c_n x^n$ has a radius of convergence R, then

a.) $f'(x) = \sum_{n=1}^{\infty} c_n n x^{n-1}$ and has a radius of convergence R.

b.) $\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{n+1}$ and has a radius of convergence R.

EXAMPLE 2: Express the following functions as a power series. Include the radius of convergence.

a.) $f(x) = \dfrac{1}{(2+x)^2}$

b.) $f(x) = \dfrac{x^2}{(1-2x)^2}$

c.) $f(x) = \ln(3 + x)$

d.) Use part c.) to find a power series representation for $f(x) = x\ln(3 + x)$

e.) $f(x) = \arctan x$

f.) Use part d.) to find a power series representation for $f(x) = \arctan(x^3)$

EXAMPLE 3: Evaluate $\int_0^{1/2} \frac{1}{1+x^6}\,dx$.

EXAMPLE 4: Approximate $\int_0^{0.1} \frac{1}{1+x^3}\,dx$ with error less than $\frac{1}{100}$.