Section 10.7: Taylor and Maclaurin Series

The construction of a Taylor Series:

Let $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$

$$= c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + c_5(x-a)^5 + ... + c_i(x-1)^i + ...$$

Substituting $x = a$ into $f(x)$ gives $\boxed{c_0 = f(a)}$

Take the derivative of $f(x)$:

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + 5c_5(x-a)^4 + ...$$

Substituting $x = a$ into $f'(x)$ gives $\boxed{c_1 = f'(a)}$

Likewise,

$$f''(x) = 2c_2 + 2 \cdot 3c_3(x-a) + 3 \cdot 4c_4(x-a)^2 + 4 \cdot 5c_5(x-a)^3 + ...$$

Substituting $x = a$ into $f''(x)$ gives $f''(a) = 2c_2$, yielding $\boxed{c_2 = \frac{f''(a)}{2}}$

$$f'''(x) = 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4(x-a) + 3 \cdot 4 \cdot 5c_5(x-a)^2 + ...$$

Substituting $x = a$ into $f'''(x)$ gives $f'''(a) = 2 \cdot 3c_3 = 3!c_3$, yielding $\boxed{c_3 = \frac{f'''(a)}{3!}}$

Continuing in this matter, we find $\boxed{c_i = \frac{f^i(a)}{i!}}$

Thus, we define the **Taylor Series** for $f(x)$ about $x = a$ to be

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + ...$$

where $f^{(n)}(a)$ is the nth derivative of $f(x)$ at $x = a$.

EXAMPLE 1: Find the Taylor Series for $f(x) = e^{2x}$ at $x = 1$.

EXAMPLE 2: Find the Taylor Series for $f(x) = \ln x$ at $x = 2$.

Definition: The Maclaurin Series for the function $f(x)$ is defined to be the Taylor Series about $x = 0$. That is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

where $f^{(n)}(0)$ is the nth derivative of $f(x)$ at $x = 0$.

EXAMPLE 3: Find the Maclaurin Series for $f(x) = e^x$.

EXAMPLE 4: Find the Maclaurin Series for $f(x) = \cos x$.

EXAMPLE 5: Find the Maclaurin Series for $f(x) = \sin x$.

EXAMPLE 6: Find the Maclaurin Series for $f(x) = e^{x^2}$.

EXAMPLE 7: Find the Maclaurin Series for $f(x) = x\cos\frac{x}{2}$.

EXAMPLE 8: Find the Maclaurin Series for $f(x) = \frac{\sin x}{x^2}$.

EXAMPLE 9: Consider $\int_0^{1/2} \cos(x^2)\,dx$.

(a) Evaluate the integral as a series.

(b) Find the first three terms of the series found in part (a).
How accurate is this approximation to $\int_0^{1/2} \cos(x^2)\,dx$?

EXAMPLE 10: Approximate $\int_0^1 e^{-x}\, dx$ with error less than .001.

EXAMPLE 11: Find the limit by writing $\sin x$ as a power series:

$$\lim_{x\to 0}\frac{\sin x - x + \frac{1}{6}x^3}{x^5}$$

EXAMPLE 12: Consider $\frac{e^x - 1 - x}{x^2}$. This can be written as a series, using the known Maclaurin series for e^x.

a.) Write out the first 4 nonzero terms of this series.

b.) Using part a.), approximate $\int_0^2 \frac{e^x - 1 - x}{x^2}\, dx$.