Section 10.9: Taylor Polynomials

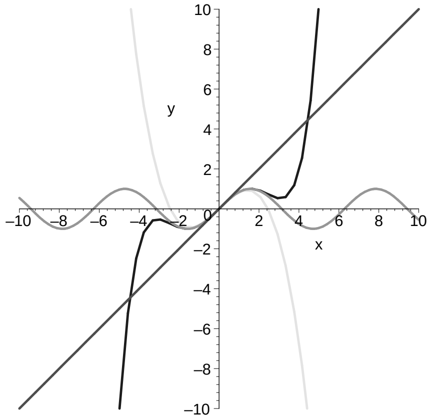
Definition: Let $f(x)$ be a function. The n^{th} degree **Taylor Polynomial** for $f(x)$ at $x = a$ is

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

where $f^{(i)}(a)$ is the ith derivative of $f(x)$ at $x = a$.

EXAMPLE 1: Find the first degree Taylor Polynomial for $f(x) = \sqrt{x}$ at $x = 1$.

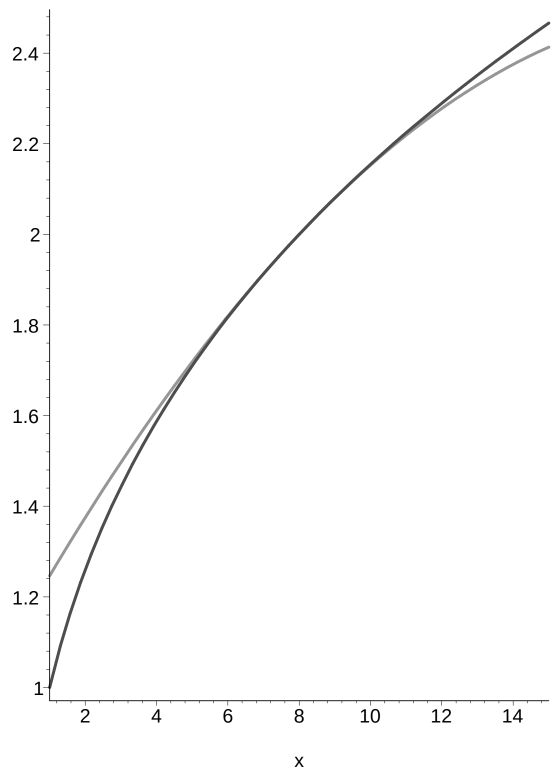
EXAMPLE 2: Find $T_1(x)$, $T_3(x)$, and $T_5(x)$ for $f(x) = \sin x$ at $x = 0$. Notice by viewing the graph below how the Taylor Polynomials better approximate $f(x)$ at $x = 0$ as n gets larger.



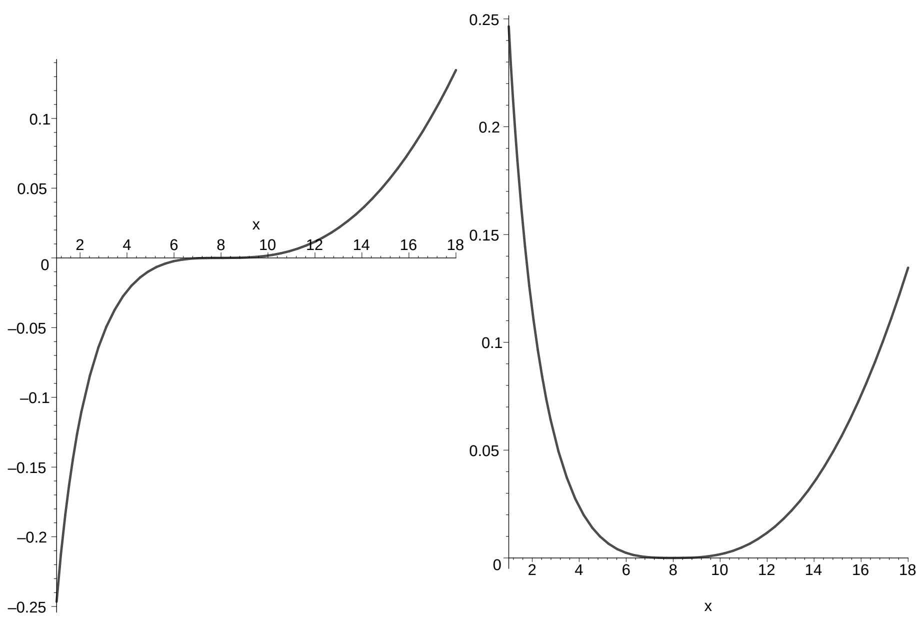
EXAMPLE 3: If $f(x) = x^{1/3}$, find $T_2(x)$ at $x = 8$.

Definition: The **Remainder** in using $T_n(x)$ to approximate $f(x)$ for x near a is defined to be $R_n(x) = f(x) - T_n(x)$. Below, you will see a graph of the function and the Taylor Polynomial we found in example 3. Note that the function and the Taylor Polynomial are the same at $x = a$ (where $a = 8$ in this example), and as x deviates from $x = 8$, the Taylor Polynomial deviates from $f(x)$. You will also see the graph of the remainder and the graph of the absolute value of the remainder.

Second degree Taylor Polynomial for $f(x) = x^{1/3}$ at $x = 8$.



Remainder and absolute value of remainder:



Taylor's Inequality: An upper bound on the absolute value of the remainder is

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

where $|f^{(n+1)}(x)| \leq M$ for x in an interval containing a.

EXAMPLE 4: If $f(x) = \ln x$ and $n = 3$, find $T_n(x)$ at $x = 4$. Then, use Taylor's inequality to find an upper bound on the remainder if $3 \leq x \leq 5$.

EXAMPLE 5: If $f(x) = \cos x$, $n = 2$. Find $T_n(x)$ at $x = \dfrac{\pi}{4}$. Then use Taylor's inequality to find an upper bound on the remainder if $\dfrac{\pi}{6} \le x \le \dfrac{2\pi}{3}$.

EXAMPLE 6: Recall $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ is the Maclaurin Series for $\sin x$.

(a) What is the maximum error possible in using the approximation $\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$ for $|x| < 0.3$?

(b) Use $\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$ to approximate $\sin(1.2^\circ)$.