Section 11.3: Cross Product

It is desirable to be able to construct a perpendicular vector from two existing vectors, and the cross product provides a means for doing so. More precisely, the **Cross Product** of two vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ is given by

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

In order to make this expression for $\mathbf{a} \times \mathbf{b}$ easier to remember, we will use the notation of determinates. A determinate of order 2 is defined by

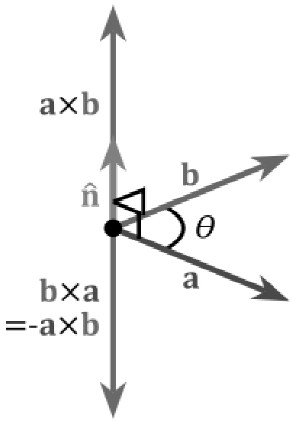
$$\begin{vmatrix} \text{a} & \text{b} \\ \text{c} & \text{d} \end{vmatrix} = ad - bc$$

A determinate of order 3 is defined by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

EXAMPLE 1: Find the cross product of $\langle 1, 1, 3 \rangle$ and $\langle -2, -1, -5 \rangle$.

Theorem: The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.



EXAMPLE 2: Find a unit vector perpendicular to both $\langle 1, 2, 1\rangle$ and $\langle 0, 1, 3\rangle$.

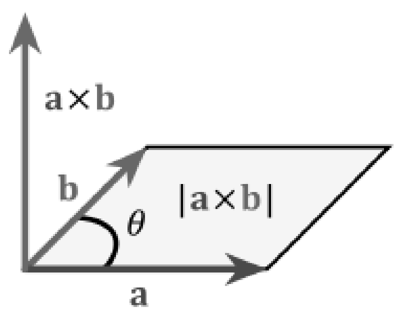
EXAMPLE 3: Find a vector perpendicular to the plane through the points $P(2, 3, 5)$, $Q(-1, 3, 4)$ and $R(3, 0, 6)$.

Theorem: If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, then $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$.

Corollary: Two non zero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if $|\mathbf{a} \times \mathbf{b}| = 0$.

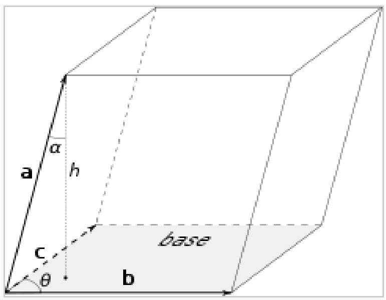
EXAMPLE 4: Determine whether the vectors are parallel: $\mathbf{a} = 2\mathbf{i} + 6\mathbf{j} - 4\mathbf{k}$ and $\mathbf{b} = -3\mathbf{i} - 9\mathbf{j} + 6\mathbf{k}$.

Theorem: The length of the cross product $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$.



EXAMPLE 5: Find the area of the triangle determined by the points $P(1, 0, 0)$, $Q(0, 2, 0)$ and $R(0, 0, 3)$.

Scalar Triple Product The product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is called the scalar triple product of the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. The geometric significance of the scalar triple product is the construction of the parallelipiped determined by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.



Vectors defining a parallelepiped.

EXAMPLE 6: Find the volume of the parallelipiped determined by the vectors $\langle 2, 3, -2 \rangle$, $\langle 1, -1, 0 \rangle$ and $\langle 2, 0, 3 \rangle$.

EXAMPLE 7: State whether each expression is meaningful. If not, explain why. If so, state whether the expression is a vector or a scalar.

a.) $\mathbf{a} \cdot \mathbf{b}$

b.) $\mathbf{a} \times \mathbf{b}$

c.) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$

d.) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$

e.) $(\mathbf{a} \cdot \mathbf{b}) \times (\mathbf{c} \cdot \mathbf{d})$

f.) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$

g.) $\mathbf{a} \times (\mathbf{b} \cdot \mathbf{c})$

h.) $|\mathbf{a}|(\mathbf{b} \times \mathbf{c})$