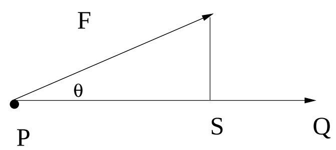
Section 1.2: Dot Product

Definition: Recall that the **Work** W done by a constant force F in moving an object through a distance d is $W = Fd$. However this formula only applies when the force is applied in the direction of motion. Suppose now we have an object moving from the point P to the point Q under a force F as shown.



Then the work W done in moving the object from P to Q depends on two things:
(a) The distance the object has moved, namely $|\vec{D}| = |\vec{PQ}|$ and
(b) The magnitude of the force applied in the direction of motion, that is $|\vec{PS}| = |\vec{F}|\cos(\theta)$. Thus the work W done in moving the object is given by

$$W = |\vec{F}||\vec{D}|\cos(\theta)$$

EXAMPLE 1: Find the work done by a force of 20 lbs acting in the direction $N50°W$ in moving an object 4 feet due west.

Definition: The dot product of the vectors $\vec{a}$ and $\vec{b}$ is defined as $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos(\theta)$, where θ is the angle between $\vec{a}$ and $\vec{b}$.

EXAMPLE 2: Find the dot product of the vectors $\vec{a}$ and $\vec{b}$ if it is known that $\vec{a}$ is a unit vector, $|\vec{b}| = 5$ and $\theta = 30°$.

Theorem: $\vec{a} \cdot \vec{b} = \langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1 b_1 + a_2 b_2$. This can be proved using the Law of Cosines, which can be found on page 56 of the Stewart Text.

Note:

(a) $\vec{a}$ is perpendicular to $\vec{b}$ if $\theta = 90°$. How can we tell if two vectors are perpendicular?

(b) $\vec{a}$ is parallel to $\vec{b}$ if $\theta = 0°$ or $\theta = 180°$. How can we tell if two vectors are parallel?

EXAMPLE 3: Determine whether the vectors are parallel, perpendicular, or neither:

(a) $\langle 2, 6 \rangle$ and $\langle -3, 1 \rangle$

(b) $\langle 2, 6 \rangle$ and $\langle 3, 9 \rangle$

(c) $\langle -1, 4 \rangle$ and $\langle 2, 7 \rangle$

EXAMPLE 4: Find the dot product of the vectors $4\vec{i} + \vec{j}$ and $-3\vec{j}$.

EXAMPLE 5: Using the formula $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos(\theta)$, find the angle between the vectors $\langle 1, 5 \rangle$ and $\langle -2, 3 \rangle$.

EXAMPLE 6: A force with representation $\vec{F} = \langle 3, 8 \rangle$ moves an object along a straight line from the point $(2, 3)$ to the point $(4, 5)$. Find the work done if the distance is measured in meters and the magnitude of the force is measured in Newtons.

Definition: If $\vec{a} = \langle a_1, a_2 \rangle$, then the *orthogonal complement* of $\vec{a}$ is $\vec{a}^{\perp} = \langle -a_2, a_1 \rangle$. $\vec{a}^{\perp}$ is perpendicular to $\vec{a}$ and has the same length as $\vec{a}$. Note: There is a second vector that is orthogonal to $\vec{a}$, namely $\langle a_2, -a_1 \rangle$, it just does not have a special name.

EXAMPLE 7: Find the orthogonal complement of $\vec{a} = \langle -1, 4 \rangle$. Graph both $\vec{a}$ and $\vec{a}^{\perp}$ on the same axis.

EXAMPLE 8: Find two unit vectors perpendicular to $\langle 2, -3 \rangle$.

Vector and Scalar Projections: Given $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, we want to project $\mathbf{b}$ onto $\mathbf{a}$.

- The **Scalar Projection** of **b** onto **a** (also called the component of **b** onto **a**) is:

$$\mathbf{comp_a b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$$

- The **Vector Projection** of **b** onto **a** is:

$$\mathbf{proj_a b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}$$

EXAMPLE 9: Find the vector and scalar projection of $\langle 4, 8 \rangle$ onto $\langle 2, 1 \rangle$.

EXAMPLE 10: Find the length of the vector projection of $\langle 2, 1 \rangle$ onto $\langle -5, 1 \rangle$.

EXAMPLE 11: Find the distance from the point $P(2,1)$ to the line $y = 2x + 1$.