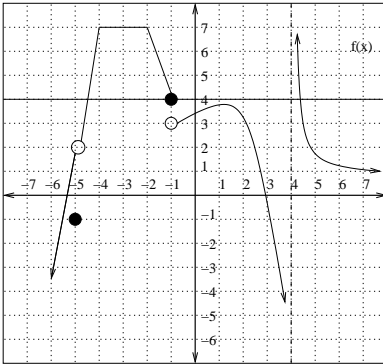


## Section 2.5: Continuity

**Definition** We say  $f(x)$  is continuous at  $x = a$  if  $\lim_{x \rightarrow a} f(x) = f(a)$ . Note that in order for this definition to be met, the following conditions must hold:

- $x = a$  is in the domain of  $f(x)$  (This ensures that  $f(a)$  exists).
- $\lim_{x \rightarrow a} f(x)$  must exist.
- $\lim_{x \rightarrow a} f(x) = f(a)$ .

*EXAMPLE 1:* For the graph of  $f(x)$  given below, locate all discontinuities. For each discontinuity, find the limit from the left and the limit from the right.



*EXAMPLE 2:* Explain why the following functions are not continuous at the indicated value of  $x$ :

(i)  $f(x) = \frac{-1}{(1-x)^2}$ ,  $x = 1$ .

(ii)  $f(x) = \begin{cases} \frac{x^2 - 2x - 8}{x - 4} & \text{if } x \neq 4 \\ 3 & \text{if } x = 4 \end{cases}$

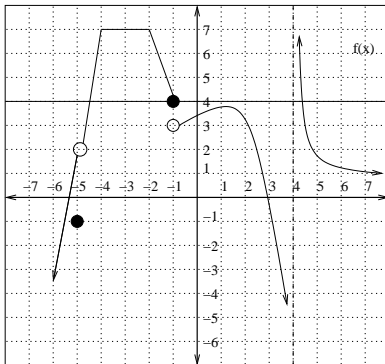
*EXAMPLE 3:* Find all points of discontinuity for  $f(x) = \begin{cases} 2x + 1 & \text{if } x \leq -1 \\ 3x & \text{if } -1 < x < 1 \\ x^2 + 2 & \text{if } x > 1 \\ 4 & \text{if } x = 1 \end{cases}$

*EXAMPLE 4:* If  $g(x) = \begin{cases} x^2 - c^2 & \text{if } x < 4 \\ cx + 20 & \text{if } x \geq 4 \end{cases}$  For what value(s) of  $c$  is  $g(x)$  continuous? For this value of  $c$ , show  $\lim_{x \rightarrow 4} g(x) = g(4)$ .

*EXAMPLE 5:* If  $f(x) = \begin{cases} x^2 + c & \text{if } x > 1 \\ 4 & \text{if } x = 1 \\ 4cx + 3 & \text{if } x < 1 \end{cases}$ , For what value(s) of  $c$  is  $f(x)$  continuous, if any?

**Definition** We say  $f(x)$  is continuous from the right at  $x = a$  if  $\lim_{x \rightarrow a^+} f(x) = f(a)$ . Similarly, we say  $f(x)$  is continuous from the left at  $x = a$  if  $\lim_{x \rightarrow a^-} f(x) = f(a)$ .

*EXAMPLE 6:* Going back to the graph of  $f(x)$  in example 1, for each point of discontinuity, determine whether  $f(x)$  is continuous from the right, left, or neither.



*EXAMPLE 7:* Which of the following functions has removable discontinuity at  $x = a$ ? If the discontinuity is removable, find a function  $g$  that agrees with  $f$  for  $x \neq a$  and is continuous at  $x = a$ . Note:  $f$  has removable discontinuity at  $x = a$  if  $\lim_{x \rightarrow a} f(x)$  exists and  $f(x)$  can be redefined so that  $\lim_{x \rightarrow a} f(x) = f(a)$  (thereby removing the discontinuity).

(a)  $f(x) = \frac{x^4 - 16}{x - 2}, x = 2.$

(b)  $f(x) = \frac{1}{x - 1}, x = 1$

(c)  $f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ 2x + 4 & \text{if } x \geq 1 \end{cases}, x = 1$

**Intermediate Value Theorem** If  $f(x)$  is continuous on the interval  $[a, b]$  and  $N$  is any number strictly between  $f(a)$  and  $f(b)$ , then there is a number  $c$ ,  $a < c < b$ , so that  $f(c) = N$ .

*EXAMPLE 8:* If  $g(x) = x^5 - 2x^3 + x^2 + 2$ , show there a number  $c$  so that  $g(c) = -1$ .

*EXAMPLE 9:* Show there is a root to the equation  $x^5 - 2x^4 - x - 3$  on the interval  $(2, 3)$ .