Section 2.5: Continuity

**Definition** We say \( f(x) \) is continuous at \( x = a \) if \( \lim_{x \to a} f(x) = f(a) \). Note that in order for this definition to be met, the following conditions must hold:

(a) \( x = a \) is in the domain of \( f(x) \) (This ensures that \( f(a) \) is defined).

(b) \( \lim_{x \to a} f(x) \) must exist. Thus \( \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) \).

(c) \( \lim_{x \to a} f(x) = f(a) \).

Some types of discontinuities include ‘holes’, ‘jumps’ and ‘vertical asymptotes’.

The graphs below represents a discontinuity at \( x = 4 \) because \( f(4) \) is not defined.

The graphs below represents a discontinuity at \( x = 4 \) because \( \lim_{x \to 4} f(x) \) does not exist.

The graphs below represents a discontinuity at \( x = 4 \) because \( \lim_{x \to 4} f(x) \neq f(4) \).
EXAMPLE 1: For the graph of $f(x)$ given below, locate all discontinuities. For each discontinuity, find the limit from the left and the limit from the right.

EXAMPLE 2: Explain why the following functions are or are not continuous at the indicated value of $x$. Support your answer.

(i) $f(x) = \frac{-1}{(1-x)^2}$, $x = 1$.

(ii) $f(x) = \begin{cases} \frac{x^2 - 2x - 8}{x - 4} & \text{if } x \neq 4 \\ 3 & \text{if } x = 4 \end{cases}$ at $x = 4$

(iii) $f(x) = \begin{cases} 2x + 1 & \text{if } x \leq -1 \\ 3x & \text{if } -1 < x < 1 \\ x^2 + 2 & \text{if } x > 1 \\ 4 & \text{if } x = 1 \end{cases}$ at $x = \pm 1$
EXAMPLE 3: If \( g(x) = \begin{cases} x^2 - c^2 & \text{if } x < 4 \\ cx + 20 & \text{if } x \geq 4 \end{cases} \) For what value(s) of \( c \) is \( g(x) \) continuous? In order to receive full credit, your answer must be fully supported by the definition of continuity.

EXAMPLE 4: If \( f(x) = \begin{cases} x^2 + c & \text{if } x > 1 \\ 4 & \text{if } x = 1 \\ 4cx + 3 & \text{if } x < 1 \end{cases} \), For what value(s) of \( c \) is \( f(x) \) continuous, if any? In order to receive full credit, your answer must be fully supported by the definition of continuity.

**Definition** We say \( f(x) \) is continuous from the right at \( x = a \) if \( \lim_{x \to a^+} f(x) = f(a) \). Similarly, we say \( f(x) \) is continuous from the left at \( x = a \) if \( \lim_{x \to a^-} f(x) = f(a) \).
**Intermediate Value Theorem** If $f(x)$ is continuous on the interval $[a, b]$ and $N$ is any number strictly between $f(a)$ and $f(b)$, then there is a number $c$, $a < c < b$, so that $f(c) = N$.

**EXAMPLE 5:** If $g(x) = x^5 - 2x^3 + x^2 + 2$, show there is a number $c$ so that $g(c) = -1$.

**EXAMPLE 6:** Show there is a root to the equation $x^5 - 2x^4 - x - 3$ on the interval $(2, 3)$. 