

## Section 3.11: Differentials; Linear and Quadratic Approximations

**Differentials:** Let  $y = f(x)$ . If  $x$  changes from  $x_1$  to  $x_2$ , then the change in  $x$  is

$$\Delta x = x_2 - x_1$$

The *corresponding* change in  $y$  is

$$\Delta y = f(x_2) - f(x_1)$$

*EXAMPLE 1:* Let  $y = 4 - x^2$ . Find  $\Delta y$  if  $x$  changes from  $x = 1$  to  $x = 1.5$ . Illustrate these quantities graphically.

**Definition:** If  $y = f(x)$ , the **differential**  $dx$  is an independent variable. The **differential**  $dy$  depends on  $dx$  and is defined as

$$dy = f'(x)dx$$

*EXAMPLE 2:* If  $f(x) = 4 - x^2$ , find  $dy$  if  $x = 1$  and  $dx = \frac{1}{2}$ .

*EXAMPLE 3:* Find the differential of  $y = u \tan u$ .

*EXAMPLE 4:* Find the differential  $dy$  if  $y = \frac{r}{r+1}$  and  $dr = 0.5$ .

**How we use differentials:** If  $\Delta x$  is small, then  $\Delta y \approx dy$ . We can use this fact to approximate a function as follows:

Suppose  $x$  changes from  $x = a$  to  $x = a + \Delta x$ . Then

$$\Delta y = f(a + \Delta x) - f(a)$$

$$f(a + \Delta x) = f(a) + \Delta y$$

$$f(a + \Delta x) \approx f(a) + dy \text{ (since } \Delta y \approx dy)$$

$$f(a + \Delta x) \approx f(a) + f'(a)dx$$

*EXAMPLE 5:* Use differentials to approximate  $(1.97)^6$ .

*EXAMPLE 6:* Use differentials to approximate  $\cos(31.5^\circ)$

**Linear Approximation:** The **Linearization** or **Linear Approximation** of  $f(x)$  at  $x = a$  is the equation of the tangent line to the graph of  $f(x)$  at  $x = a$ , that is

$$L(x) = f(a) + f'(a)(x - a)$$

*EXAMPLE 7:* Find the linear approximation for  $f(x) = \frac{1}{x}$  at  $x = 4$ .

*EXAMPLE 8:* Find the linear approximation for  $f(x) = \sqrt[3]{x+1}$  at  $x = 0$  and use it to approximate  $\sqrt[3]{0.95}$

*EXAMPLE 9:* Find the linear approximation for  $f(x) = \sin x$  at  $x = 0$  and use it to approximate  $\sin(0.1)$

*EXAMPLE 10:* The radius of a circular disk is given to be 24 cm with a maximum error in measurement of 0.2 cm.

(a) What is the maximum error in the calculated area of the disk?

(b) Use differentials to approximate the maximum error in the area of the disk.

(c) What is the relative error?

*EXAMPLE 11:* Suppose that we don't have a formula for  $g(x)$  but we know that  $g(2) = 4$  and  $g'(x) = \sqrt{x^2 + 5}$  for all  $x$ . Use a linear approximation to estimate  $g(1.95)$  and  $g(2.05)$ .

**Quadratic Approximation:** The Quadratic Approximation for a function  $f(x)$  at  $x = a$  is

$$Q(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2$$

*EXAMPLE 12:* Find the quadratic approximation for  $f(x) = \cos x$  at  $x = 0$ .