Section 3.2: Differentiation Formulas

Differentiation Formulas:

(1) Constant rule: If $f(x) = c$, where c is a constant, then $f'(x) = 0$.

(2) Power rule: If $f(x) = x^n$, then $f'(x) = nx^{n-1}$

(3) Constant times a function rule: $\frac{d}{dx} cf(x) = c\frac{d}{dx} f(x)$

(4) Sum/Difference rule: If $f(x) = g(x) \pm h(x)$, then $f'(x) = g'(x) \pm h'(x)$

(5) Product rule: If $f(x) = g(x)h(x)$, then $f'(x) = g(x)h'(x) + g'(x)h(x)$

(6) Quotient rule: If $f(x) = \frac{g(x)}{h(x)}$, then $f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{(h(x))^2}$

EXAMPLE 1: Find the derivative of the following functions.

(a) $g(x) = x^5 + 8x^2 - 16x + 2 - \pi^2$

$g'(x) = 5x^4 + 16x - 16$

(b) $f(t) = (1 - \sqrt{t})^2$ $= (1-\sqrt{t})(1-\sqrt{t})$

$f(t) = 1 - 2\sqrt{t} + t = 1 - 2t^{\frac{1}{2}} + t$

$f'(t) = 0 - t^{-\frac{1}{2}} + 1$

$f'(t) = -\frac{1}{\sqrt{t}} + 1$

(c) $H(s) = \left(\frac{s}{2}\right)^5$ $= \frac{1}{32} s^5$

$H'(s) = \frac{5}{32} s^4$

(d) $F(x) = \frac{x - 3x\sqrt{x}}{\sqrt{x}} = \frac{x}{\sqrt{x}} - \frac{3x\cancel{\sqrt{x}}}{\cancel{\sqrt{x}}}$

$$\frac{x}{\sqrt{x}} = \frac{x^1}{x^{\frac{1}{2}}} = x^{\frac{1}{2}}$$

$$F(x) = x^{\frac{1}{2}} - 3x$$

$$F'(x) = \frac{1}{2}x^{-\frac{1}{2}} - 3 = \frac{1}{2\sqrt{x}} - 3$$

(e) $y = \underbrace{(x^3 - x^2 - 2x + 1)}_{f}\underbrace{(5x^4 - 20x^3 + 5x + 3)}_{g}$

$$y' = fg' + f'g$$

$$y' = \underbrace{(x^3 - x^2 - 2x + 1)}_{f}\underbrace{(20x^3 - 60x^2 + 5)}_{g'} + \underbrace{(3x^2 - 2x - 2)}_{f'}\underbrace{(5x^4 - 20x^3 + 5x + 3)}_{g}$$

(f) $f(u) = \frac{1 - u^2}{1 + u^2}$ $\frac{g}{h}$

$$f'(u) = \frac{g'h - gh'}{h^2} = \frac{\overbrace{(-2u)}^{g'}\overbrace{(1+u^2)}^{h} - \overbrace{(1-u^2)}^{g}\overbrace{(2u)}^{h'}}{(1+u^2)^2}$$

EXAMPLE 2: If $f(5) = 1$, $f'(5) = 6$, $g(5) = -3$ and $g'(5) = 2$, find the value of $(fg)'(5)$.

$$(fg)'(5) = f(5)g'(5) + f'(5)g(5)$$
$$= (1)(2) + (6)(-3)$$
$$= 2 - 18 = \boxed{-16}$$

EXAMPLE 3: Find the equation of the tangent line to the graph of $f(x) = x + \sqrt{x}$ at the point $(1, 2)$. ← point

$$f(x) = x + x^{\frac{1}{2}}$$
$$f'(x) = 1 + \tfrac{1}{2}x^{-\frac{1}{2}}$$
$$f'(x) = 1 + \frac{1}{2\sqrt{x}}$$

$$m = f'(1)$$
$$m = 1 + \frac{1}{2\sqrt{1}} = 1 + \frac{1}{2} = \frac{3}{2} \leftarrow m$$

$$\boxed{y - 2 = \frac{3}{2}(x - 1)}$$

$f(x) = x\sqrt{x}$

EXAMPLE 4: At what point on the curve $y = x\sqrt{x}$ is the tangent line parallel to the line $3x - y + 6 = 0$?

↳ same slope

① Find the slope of $3x - y + 6 = 0$

$$y = 3x + 6$$
$$\boxed{m = 3}$$

② Find the value of x so that $f'(x) = 3$

$$f(x) = x\sqrt{x} = x^{\frac{3}{2}}$$
$$f'(x) = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$$

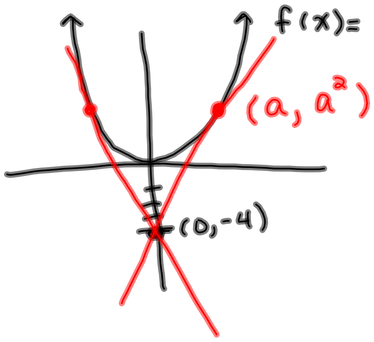
$$f'(x) = 3 \rightarrow \frac{3}{2}\sqrt{x} = 3$$
$$\sqrt{x} = 3 \cdot \frac{2}{3}$$
$$\sqrt{x} = 2$$
$$x = 4$$

$$f(4) = 4\sqrt{4} = 8$$

$$\boxed{\text{Point} = (4, 8)}$$

EXAMPLE 5: Show there are two tangent lines to the parabola $y = x^2$ that pass through the point $(0, -4)$. Find the equation of these tangent lines.

$f(x) = x^2 \rightarrow f'(x) = 2x$



① line is tangent to the parabola at $x = a$.

$m = f'(a)$

$\boxed{m = 2a}$

② line passes thru (a, a^2) and $(0,-4)$

$m = \dfrac{a^2+4}{a-0}$

$\boxed{m = \dfrac{a^2+4}{a}}$

equate

$2a = \dfrac{a^2+4}{a}$

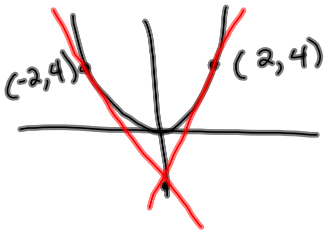
$2a^2 = a^2 + 4$

$a^2 = 4 \rightarrow a = \pm 2$

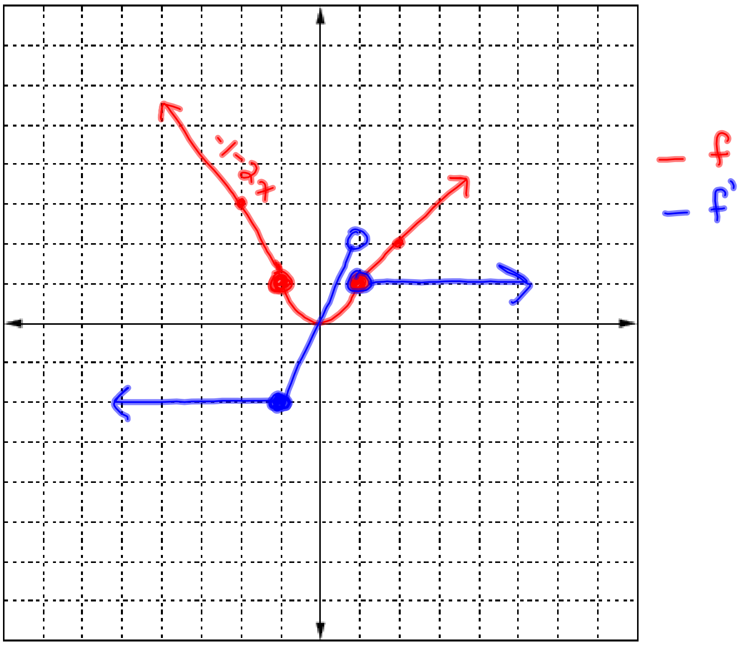
Find the equations of the two tangent lines. $f(x) = x^2$ $f'(x) = 2x$

point $(2,4)$ $m = 4$ $\boxed{y - 4 = 4(x-2)}$

point $(-2,4)$ $m = -4$ $\boxed{y - 4 = -4(x+2)}$



(ii) Sketch the graph of $f(x)$ and $f'(x)$ on the same axis.



$$f(x) = \begin{cases} -1 - 2x & \text{if } x < -1 \\ x^2 & \text{if } -1 \leq x < 1 \\ x & \text{if } x \geq 1 \end{cases}$$

$f(x)$ is not differentiable.

$$f'(x) = \begin{cases} -2 & \text{if } x < -1 \\ 2x & \text{if } -1 \leq x < 1 \\ 1 & \text{if } x > 1 \end{cases}$$

EXAMPLE 7: If $f(x) = \begin{cases} x^2 & \text{if } x \le 2 \\ mx+b & \text{if } x > 2 \end{cases}$, find the value of m and b that make $f(x)$ differentiable everywhere.

① $f(x)$ must be continuous at $x=2$.

$$\lim_{x\to 2^+} f(x) = 2m+b$$

$$\lim_{x\to 2^-} f(x) = 4$$

also $f(2)$

equate these

$$\boxed{2m+b=4}$$

② $f(x)$ must be differentiable at $x=2$.

$$\lim_{x\to 2^+} f'(x) = \lim_{x\to 2^-} f'(x)$$

$$f'(x) = \begin{cases} 2x & \text{if } x \le 2 \\ m & \text{if } x > 2 \end{cases}$$

$$\boxed{m = 4}$$

$$2(4)+b=4$$

$$\boxed{b=-4}$$

EXAMPLE 8: If $\vec{r}(t) = \langle t^2+2t, t^3+3t^2 \rangle$ is the position of a moving object at time t, where the position is measured in feet and the time in seconds, find the velocity and speed at time $t = 1$

$$v(t) = r'(t) = \langle 2t+2, 3t^2+6t \rangle$$

$$\boxed{v(1) = r'(1) = \langle 4, 9 \rangle}$$

$$\text{speed} = |v(1)| = |\langle 4,9\rangle| = \sqrt{16+81}$$

$$= \sqrt{97} \ \frac{f}{s}$$