Section 3.4: Derivatives of Trigonometric Functions

Two special limits:

(1) $\lim_{x \to 0} \frac{\sin x}{x} = 1$ (2) $\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$

EXAMPLE 1: Find the limit:

(i) $\lim_{x \to 0} \frac{\sin x}{3x} = \frac{1}{3} \lim_{x \to 0} \frac{\sin x}{x}$

$= \frac{1}{3}(1)$

$= \boxed{\frac{1}{3}}$

(ii) $\lim_{x \to 0} \frac{\sin 9x}{7x} = 9 \, \frac{1}{7} \lim_{x \to 0} \frac{\sin(9x)}{9x}$

$= \frac{9}{7} \lim_{x \to 0} \frac{\sin(9x)}{9x} = \frac{9}{7}(1) = \boxed{\frac{9}{7}}$

(iii) $\lim_{x \to 0} \frac{\sin 8x}{\sin 7x} = \lim_{x \to 0} \frac{8\left(\frac{\sin 8x}{8x}\right)}{7\left(\frac{\sin 7x}{7x}\right)} = \frac{8}{7} \lim_{x \to 0} \frac{\cancel{\frac{\sin 8x}{8x}} \to 1}{\cancel{\frac{\sin 7x}{7x}} \to 1}$

$= \boxed{\frac{8}{7}}$

ex)

$\lim_{x \to \frac{\pi}{4}} \frac{\sin x}{x} = \frac{\frac{\sqrt{2}}{2}}{\frac{\pi}{4}}$

(iv) $\lim_{x\to0}\frac{\tan^2 4x}{x^2} = \lim_{x\to0}\left(\frac{\tan 4x}{x}\right)\left(\frac{\tan 4x}{x}\right)$

Know: $\lim_{x\to0}\frac{\sin 4x}{4x} = 1$

$$\lim_{x\to0}\left(\frac{\sin 4x}{(\cos 4x)(x)}\right)\left(\frac{\sin 4x}{(\cos 4x)(x)}\right)$$

$$16\lim_{x\to0}\left(\cancel{\frac{\sin 4x}{4x}}^{\to1}\cdot\cancel{\frac{1}{\cos 4x}}^{\to1}\right)\left(\cancel{\frac{\sin 4x}{4x}}^{\to1}\ \cancel{\frac{1}{\cos 4x}}^{\to1}\right) = \boxed{16}$$

$\uparrow$ $\cos 0 = 1$

(v) $\lim_{x\to0}\frac{\cos x - 1}{\sin x} = \lim_{x\to0}\frac{\frac{\cos x - 1}{x}}{\frac{\sin x}{x}} = \frac{0}{1} = \boxed{0}$

<u>Derivatives of Trigonometric Functions:</u>

Function	Derivative
* $\sin x$	$\cos x$
* $\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\csc^2 x$
$\csc x$	$-\csc x \cot x$

$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x}$$

$$= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

EXAMPLE 2: Find the derivative:

(i) $f(x) = \sec^2 x + 4\tan x + x\sqrt{x}$

$$= \underbrace{\underbrace{(\sec x)}_{1}\underbrace{(\sec x)}_{2}}_{\text{product rule}} + 4\tan x + x^{\frac{3}{2}}$$

$$= \underbrace{(\sec x)}_{1}\underbrace{(\sec x\tan x)}_{2'} + \underbrace{(\sec x\tan x)}_{1'}\underbrace{(\sec x)}_{2} + 4\sec^2 x + \frac{3}{2}x^{\frac{1}{2}}$$

$$= \boxed{2\sec^2 x\tan x + 4\sec^2 x + \frac{3}{2}\sqrt{x}}$$

(ii) $g(t) = \dfrac{2\cos t + 1}{\cot t + t}$ $\quad \dfrac{T}{B}$

$$g'(t) = \frac{T'B - TB'}{B^2}$$

$$= \frac{(-2\sin t)(\cot(t) + t) - (2\cos t + 1)(-\csc^2(t) + 1)}{(\cot(t) + t)^2}$$

EXAMPLE 3: Find the equation of the tangent line to the graph of $f(x) = 2\sin x$ at $x = \frac{\pi}{3}$.

$f'(x) = 2\cos x$

$m = f'\left(\frac{\pi}{3}\right)$

$$\boxed{m = 2\cos\frac{\pi}{3} = 2\cdot\frac{1}{2} = 1}$$

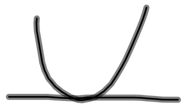
point: $\left(\frac{\pi}{3}, 2\underbrace{\sin\frac{\pi}{3}}_{\frac{\sqrt{3}}{2}}\right) = \left(\frac{\pi}{3}, \sqrt{3}\right)$

$$y - \sqrt{3} = 1\left(x - \frac{\pi}{3}\right)$$

$$\boxed{y = x - \frac{\pi}{3} + \sqrt{3}}$$

EXAMPLE 4: For what value(s) of x does the graph of $f(x) = x + 2\sin x$ have a horizontal tangent?

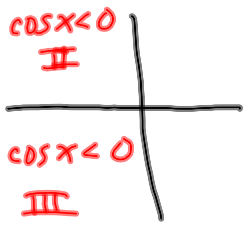
horizontal tangent $\rightarrow$ $f'(x) = 0$



$$1 + 2\cos x = 0$$

$$2\cos x = -1$$

$$\cos x = -\frac{1}{2}$$



Q II: $\cos x = -\frac{1}{2}$

$$x = \pi - \frac{\pi}{3}$$

$$\boxed{x = \frac{2\pi}{3} + 2n\pi \quad n = \text{integer}}$$

Q III: $\cos x = -\frac{1}{2}$

$$x = \pi + \frac{\pi}{3}$$

$$= \boxed{\frac{4\pi}{3} + 2n\pi}$$