Chain Rule: We use the chain rule when we are differentiating a function written as a composition of functions, that is $f(x) = g(h(x))$. Then $f'(x) = g'(h(x))h'(x)$.

$f(x) = g \circ h = g(h(x))$

ex: $g(x) = x^3$
$h(x) = \sin x$
$g \circ h = (\sin x)^3$

$\frac{d}{dx} \sin x = \cos x$
$\frac{d}{dx} \cos x = -\sin x$
$* \frac{d}{dx} \tan x = \sec^2 x$
$\frac{d}{dx} \sec x = \sec x \tan x$
$\frac{d}{dx} \cot x = -\csc^2 x$
$\frac{d}{dx} \csc x = -\csc x \cot x$

EXAMPLE 1: Find the derivative:

(i) $f(x) = \sin(2x) + \cot(5x^2)$

$f'(x) = \cos(2x) \cdot 2 + -\csc^2(5x^2) \cdot 10x$

(ii) $g(t) = \tan(\cos(t))$

$g'(t) = \sec^2(\cos t) \cdot (-\sin t)$

(iii) $h(w) = \sec(\cos(\sin(4w^2)))$
1 2 3 4

$\sec(\cos(\sin(4w^2)))\tan(\cos(\sin(4w^2))) \cdot (-\sin(\sin(4w^2))) \cdot \cos(4w^2) \cdot 8w$

Generalized Power Rule: If $f(x) = (g(x))^n$, then $f'(x) = n(g(x))^{n-1}g'(x)$

EXAMPLE 2: Find the derivative:

(i) $f(x) = \dfrac{1}{(x^2+5x+4)^{10}}$

① quotient rule

② $f(x) = (x^2+5x+4)^{-10}$

$f'(x) = -10(x^2+5x+4)^{-11} \cdot (2x+5)$

(ii) $g(x) = \underbrace{x^3}_{u}\underbrace{(\sqrt{x}+5)^3}_{v}$

$g'(x) = u'v + uv'$

$x^{\frac{1}{2}}$

$= \underbrace{3x^2}_{u'}\underbrace{(\sqrt{x}+5)^3}_{v} + x^3\left[3(\sqrt{x}+5)^2 \cdot \frac{1}{2}x^{-\frac{1}{2}}\right]$

(iii) $f(x) = \sin(3x) + \sin^3(x)$

$= \sin(3x) + (\sin x)^3$

$f'(x) = \cos(3x)\cdot 3 + 3(\sin x)^2 \cdot \cos x$

(iv) $h(t)=\sqrt{\cos(\sin^2 t)} = \left(\cos(\sin^2 t)\right)^{\frac{1}{2}}$

$\sin^2 t = (\sin t)^2$

$h'(t) = \frac{1}{2}\left(\cos(\sin^2 t)\right)^{-\frac{1}{2}}\left(-\sin(\sin^2 t)\cdot 2(\sin t)^1(\cos t)\right)$

(v) $g(x)=\sqrt{x+\sqrt{x+\sqrt{x}}}$

$\frac{d}{dt}(\sin t)^2 = 2(\sin t)(\cos t)$

$g(x) = \left(x+\left(x+x^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}$

$g'(x) = \frac{1}{2}\left(x+\left(x+x^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{-\frac{1}{2}}\left(1+\frac{1}{2}\left(x+x^{\frac{1}{2}}\right)^{-\frac{1}{2}}\cdot\left(1+\frac{1}{2}x^{-\frac{1}{2}}\right)\right)$

EXAMPLE 3: Find the equation of the tangent line to the graph of $f(x)=8\sqrt{4+3x}$ at $x=4$.

① $m = f'(4)$

$\boxed{m=3}$

$f(x) = 8(4+3x)^{\frac{1}{2}}$

$f'(x) = 4(4+3x)^{-\frac{1}{2}}(3) = \frac{12}{\sqrt{4+3x}}$

② point: $(4, 8\sqrt{16})$

$(4, 32)$

$f'(4) = \frac{12}{\sqrt{16}} = \frac{12}{4} = 3$

$$\boxed{y-32=3(x-4)}$$

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EXAMPLE 4: Suppose $w = u \circ v$ and $u(0) = 1$, $v(0) = 2$, $u'(0) = 3$, $u'(2) = 4$, $v'(0) = 5$ and $v'(2) = 6$. Find $w'(0)$.

$w = u \circ v = u(v(x))$ composition, use chain rule

$w'(x) = u'(v(x))v'(x)$

$w'(0) = u'(v(0))v'(0)$

$= u'(2)(5)$

$= 4(5) = \boxed{20}$

EXAMPLE 5: If $F(x) = f(\cos x)$, $G(x) = \cos(f(x))$ and $H(x) = [f(\sin x)]^3$, find $F'(x)$ and $G'(x)$ and $H'(x)$.

$F(x) = f(\cos x)$ composition use chain rule

$\boxed{F'(x) = f'(\cos x)(-\sin x)}$

$G(x) = \cos(f(x))$

$\boxed{G'(x) = -\sin(f(x)) \cdot f'(x)}$

$H(x) = [f(\sin x)]^3$

$H'(x) = 3[f(\sin x)]^2 \cdot (f'(\sin x) \cdot \cos x)$

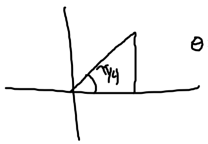
EXAMPLE 6: Find all points on the curve $y = \sin(2x) + \cos(2x)$ where the tangent line is horizontal.

$\downarrow$ $m = 0$

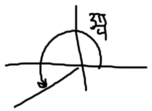
solve $y' = 0 \Rightarrow \cos(2x) \cdot 2 + -\sin(2x) \cdot 2 = 0$

$\cancel{2}\cos(2x) = \cancel{2}\sin(2x)$

$\tan\theta = 1$ QI, III



$\theta = \frac{\pi}{4} + 2n\pi$



$\theta = \frac{5\pi}{4} + 2n\pi$

$1 = \tan(2x)$

$2x = \frac{\pi}{4} + 2n\pi \Rightarrow x = \frac{\pi}{8} + n\pi$

$2x = \frac{5\pi}{4} + 2n\pi \Rightarrow x = \frac{5\pi}{8} + n\pi$