Section 3.6: Implicit Differentiation

Why we differentiate Implicitly: Suppose $y = f(x)$. In this case, we say y is an *explicit* function of x and we can therefore differentiate as usual: $\frac{dy}{dx} = f'(x)$. In this section, we investigate how to differentiate if y cannot be written as an explicit function of x, that is y is *implicitly* defined as a function of x.

EXAMPLE 1: Find $\frac{dy}{dx}$ for the following equations:

(i) $y^3 = 2x^2 + y^4$

$$3y^2 \frac{dy}{dx} = 4x + 4y^3 \frac{dy}{dx} \rightarrow 3y^2 \frac{dy}{dx} - 4y^3 \frac{dy}{dx} = 4x$$

$$\frac{dy}{dx}\left(3y^2 - 4y^3\right) = 4x$$

$$\boxed{\frac{dy}{dx} = \frac{4x}{3y^2-4y^3}}$$

(ii) $2y^2 + \underline{\underline{xy}} = x^2 + 3$

product rule

$$4y\frac{dy}{dx} + x\frac{dy}{dx} + (1)y = 2x$$

$$\frac{dy}{dx}\left(4y + x\right) = 2x - y$$

$$\boxed{\frac{dy}{dx} = \frac{2x-y}{4y+x}}$$

(iii) $\underbrace{x \sin y}_{\text{product rule}} + \underbrace{\cos(2y)}_{\text{chain rule}} = \underbrace{\cos y}_{\text{chain rule}}$

note: $y' = \frac{dy}{dx}$

$$(x)(\cos y)y' + \underbrace{(1)\sin y} + -\sin(2y)\cdot 2y' = \underbrace{(-\sin y)y'}$$

$$y'\left(x\cos y - \sin(2y)\cdot 2 + \sin y\right) = -\sin y$$

$$\boxed{y' = \frac{-\sin y}{x\cos y - 2\sin(2y) + \sin y}}$$

(iv) $\underbrace{(x^2 + y^2)^3}_{\text{chain rule}} = 2y^4 + 6x^2$

$$\underbrace{3(x^2+y^2)^2}(2x + 2yy') = 8y^3y' + 12x$$

$$\underbrace{6x(x^2+y^2)^2} + 6y(x^2+y^2)^2y' = \underbrace{8y^3y'} + 12x$$

$$y'\left(6y(x^2+y^2)^2 - 8y^3\right) = 12x - 6x(x^2+y^2)^2$$

$$\boxed{y' = \frac{12x - 6x(x^2+y^2)^2}{6y(x^2+y^2)^2 - 8y^3}}$$

EXAMPLE 2: Find the tangent line to the ellipse $\frac{x^2}{9} + \frac{y^2}{36} = 1$ at the point $(-1, 4\sqrt{2})$.

$$36\left(\frac{x^2}{9} + \frac{y^2}{36}\right) = (1)(36)$$

$$4x^2 + y^2 = 36$$

$$8x + 2yy' = 0$$

$$2yy' = -8x$$

$$y' = \frac{-4x}{y}$$

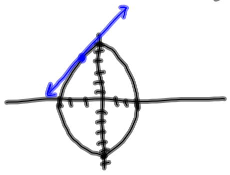
m= x=-1 + y=4√2 into y'

$$m = \frac{4}{4\sqrt{2}}$$

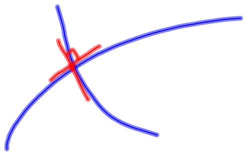
$$\boxed{m = \frac{1}{\sqrt{2}}}$$

point: $(-1, 4\sqrt{2})$

$$\boxed{y - 4\sqrt{2} = \frac{1}{\sqrt{2}}(x+1)}$$



Definition: We say two curves are orthogonal if their tangent lines are perpendicular at every point of intersection.



EXAMPLE 3: Prove the curves $x^2 - y^2 = 5$ and $4x^2 + 9y^2 = 72$ are orthogonal.

step1: intersection

$$x^2 - y^2 = 5$$

$$x^2 = 5 + y^2$$

$$4x^2 + 9y^2 = 72$$

$$4(5 + y^2) + 9y^2 = 72$$

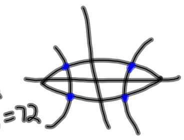
$$20 + 13y^2 = 72$$

$$13y^2 = 52$$

$$y^2 = 4$$

$y = 2$: $x = 3$, $x = -3$

$y = -2$: $x = 3$, $x = -3$



Four intersection points:
$(3,2)$, $(-3,2)$
$(3,-2)$, $(-3,-2)$

step2: show tangent lines are perpendicular at all four intersection points

Eq1: $x^2 - y^2 = 5$

$$2x - 2yy' = 0$$

$$2yy' = 2x$$

$$\boxed{y' = \frac{x}{y}}$$

Eq2: $4x^2 + 9y^2 = 72$

$$8x + 18yy' = 0$$

$$18yy' = -8x$$

$$\boxed{y' = \frac{-4x}{9y}}$$

plug in $(3,2)$

$$m = \frac{3}{2} \qquad m = \frac{-12}{18} = \frac{-2}{3}$$

tangent lines are perpendicular

By symmetry, also perp at other three points

EXAMPLE 4: If $[g(x)]^2 + 12x = x^2 g(x)$ and $g(3) = 4$, find $g'(3)$.

$$2g(x)g'(x) + 12 = x^2 g'(x) + 2x\,g(x)$$

$$g'(x)\left[2g(x) - x^2\right] = 2x\,g(x) - 12$$

$$g'(x) = \frac{2x\,g(x) - 12}{2g(x) - x^2} \qquad g'(3) = \frac{2(3)\,g(3) - 12}{2g(3) - 3^2}$$

$$= \frac{12}{8-9} = \boxed{-12}$$

EXAMPLE 5: Regard y as the independent variable and x as the dependent variable, and use implicit differentiation to find $\frac{dx}{dy}$ for the equation $(x^2 + y^2)^2 = 2x^2 y$.

chain rule — product rule

$$2(x^2+y^2)\left(2x\frac{dx}{dy} + 2y\right) = 2x^2 + 4xy\frac{dx}{dy}$$

$$4x(x^2+y^2)\frac{dx}{dy} + 4y(x^2+y^2) = 2x^2 + 4xy\frac{dx}{dy}$$

$$\frac{dx}{dy}\left(4x(x^2+y^2) - 4xy\right) = 2x^2 - 4y(x^2+y^2)$$

$$\boxed{\frac{dx}{dy} = \frac{2x^2 - 4y(x^2+y^2)}{4x(x^2+y^2) - 4xy}}$$