Section 3.7: Derivatives of Vector Functions

Sketching Vector Curves: We can sketch a vector curve by finding parametric equations, then eliminating the parameter to obtain a cartesian equation. We can determine the direction of the curve by finding the direction of a tangent vector.

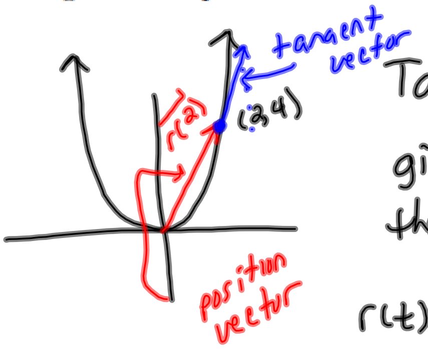
EXAMPLE 1: Sketch the following vector equations. Include the direction of the curve:

(i) $\mathbf{r}(t) = \langle t, t^2 \rangle$

$x = t$

$y = t^2$

$\boxed{y = x^2}$



Tangent vector gives the direction of the curve.

$r(t) = \langle t, t^2 \rangle$ $\quad t=2$

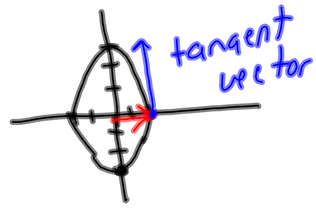
$r'(t) = \langle 1, 2t \rangle$ $\quad r'(2) = \langle 1, 4 \rangle$

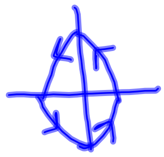
(ii) $\mathbf{r}(t) = \langle 2\cos t, 3\sin t \rangle$

$x = 2\cos t \rightarrow \frac{x}{2} = \cos t$

$y = 3\sin t \rightarrow \frac{y}{3} = \sin t$

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$





counterclockwise direction

$t = 0 \qquad r(0) = \langle 2, 0 \rangle$

$r'(t) = \langle -2\sin t, 3\cos t \rangle \qquad r'(0) = \langle 0, 3 \rangle$

$(9-t)^{\frac{1}{2}}$

EXAMPLE 2: If $\mathbf{r}(t) = \left\langle t^2 - 4, \sqrt{9-t} \right\rangle$, find the domain of $\mathbf{r}(t)$ and $\mathbf{r}'(t)$.

no domain restriction

$t \le 9$

$\boxed{\text{domain } \vec{r}(t): t \le 9}$

$$r'(t) = \left\langle 2t, \tfrac{1}{2}(9-t)^{-\frac{1}{2}}(-1) \right\rangle$$

$$= \left\langle 2t, \frac{-1}{2\sqrt{9-t}} \right\rangle$$

$\boxed{\text{domain } \vec{r}'(t): t < 9}$

EXAMPLE 3: $\mathbf{r}(t) = \langle t, 25t - 5t^2 \rangle$ is the position of a moving object at time t, where position is measured in feet and time in seconds.

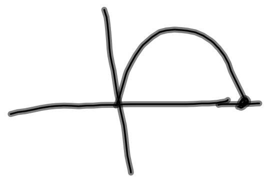
(i) Find the velocity and speed at time $t = 1$.

$v(t) = r'(t) = \langle 1, 25 - 10t \rangle$

$\boxed{v(1) = \langle 1, 15 \rangle}$ speed $= |\langle 1, 15 \rangle| = \sqrt{1 + (15)^2} = \boxed{\sqrt{226}\ \frac{f}{s}}$

(ii) With what speed does the object strike the ground?

hits ground when $y_r = 0$



$25t - 5t^2 = 0$ $\quad t = 0$

$5t(5 - t) = 0$ $\quad \boxed{t = 5}$

speed $|v(5)| = |\langle 1, -25 \rangle| = \sqrt{626}\ \frac{f}{s}$

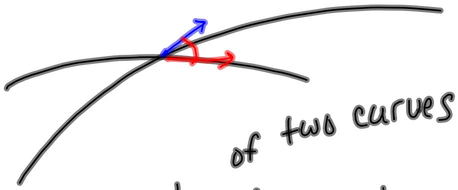
EXAMPLE 4: Find a ~~tangent vecto~~r of unit length at $t = 1$ to the curve given by $\mathbf{r}(t) = \langle t^2, 3t^3 \rangle$.

$r'(t) = \langle 2t, 9t^2 \rangle$

tangent vector $\rightarrow r'(1) = \langle 2, 9 \rangle$

unit tangent vector is $\dfrac{\langle 2, 9 \rangle}{\sqrt{4+81}} = \left\langle \dfrac{2}{\sqrt{85}}, \dfrac{9}{\sqrt{85}} \right\rangle$

EXAMPLE 5: Find the angle of intersection of the curves $\mathbf{r}(t) = \langle 1-t, 3+t^2 \rangle$ and $\mathbf{s}(u) = \langle u-2, u^2 \rangle$.



Def: The angle of intersection of two curves is the angle between the tangent vectors at the point of intersection.

① find intersection: $1-t = u-2 \rightarrow u = 3-t$

$3+t^2 = u^2$

$3+t^2 = (3-t)^2$

$3+\cancel{t^2} = 9-6t+\cancel{t^2}$

$6t = 6 \rightarrow \boxed{t=1,\ u=2}$

point of intersection $= (0,4)$

$r(t) = \langle 1-t, 3+t^2 \rangle \qquad r'(t) = \langle -1, 2t \rangle \qquad r'(1) = \langle -1, 2 \rangle$

$s(u) = \langle u-2, u^2 \rangle \qquad s'(u) = \langle 1, 2u \rangle \qquad s'(2) = \langle 1, 4 \rangle$

$$\cos\theta = \frac{\langle -1,2\rangle \cdot \langle 1,4\rangle}{\sqrt{5}\,\sqrt{17}}$$

$$\cos\theta = \frac{7}{\sqrt{5}\,\sqrt{17}} \rightarrow \boxed{\theta = \arccos\left(\frac{7}{\sqrt{5}\,\sqrt{17}}\right)}$$