Section 3.8: Higher Derivatives

Defintion: If $y = f(x)$, then the second derivative of $f(x)$ is the derivative of the first derivative. We denote the second derivative as $y'' = (f'(x))' = f''(x)$. Similarly, the third derivative is the derivative of the second derivative, denoted by $f'''(x)$. In general, the n^{th} derivative of $f(x)$ is denoted by $f^{(n)}(x)$.

EXAMPLE 1: Find the second derivative of $f(\theta) = \theta \sin(\theta)$.

$$f'(\theta) = \theta \cos(\theta) + \sin\theta$$

$$f''(\theta) = \theta(-\sin\theta) + \cos\theta + \cos\theta$$

$$\boxed{f''(\theta) = -\theta\sin\theta + 2\cos\theta}$$

EXAMPLE 2: Find the $f^{(81)}(x)$ for $f(x) = \cos(10x)$.

$$f'(x) = -\sin(10x)(10)$$

$$f''(x) = -\cos(10x)(10^2)$$

$$f'''(x) = \sin(10x)(10^3)$$

$$f^{4}(x) = \cos(10x)(10^4)$$

$$f^{80}(x) = \cos(10x)(10^{80})$$

$$f^{81}(x) = -\sin(10x)(10^{81})$$

EXAMPLE 3: Find a general formula for the n^{th} derivative for $f(x) = \dfrac{1}{x}$.

$$f = \frac{1}{x} = x^{-1}$$

$$f' = -x^{-2}$$

$$f'' = \underbrace{2}_{2!}x^{-3}$$

$$f''' = -\underbrace{3\cdot 2}_{3!}x^{-4}$$

$$f^{4} = \underbrace{4\cdot 3\cdot 2}_{4!}x^{-5}$$

$$f^{(n)} = (-1)^n n!\, x^{-(n+1)}$$

$$\boxed{f^{(n)} = \frac{(-1)^n n!}{x^{n+1}}}$$

EXAMPLE 4: If $s(t) = 2t^3 - 7t^2 + 4t + 1$ is the position of a moving object at time t, where $s(t)$ is measured in feet and t is measured in seconds, find:

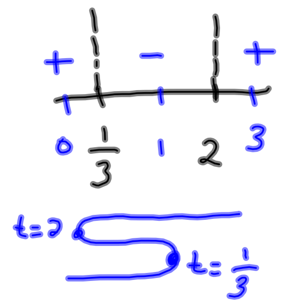
(i) The velocity at time t.

$$v(t) = s'(t) = 6t^2 - 14t + 4$$

$$= 2(3t^2 - 7t + 2)$$

$$= 2(3t-1)(t-2)$$

$$\boxed{\begin{array}{l} v(t) = 0 \\ t = \frac{1}{3},\ t = 2 \end{array}}$$



(ii) The acceleration at the times when the velocity is zero.

$$a(t) = v'(t) = s''(t)$$

$$\boxed{a(t) = 12t - 14}$$

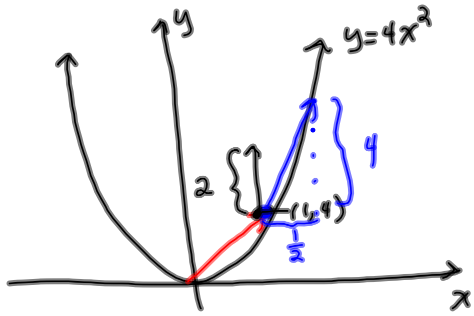
$$\boxed{\begin{array}{ll} t = \frac{1}{3}: & a\left(\frac{1}{3}\right) = 12\left(\frac{1}{3}\right) - 14 = -10\ \frac{f}{s^2} \\ t = 2: & a(2) = 12(2) - 14 = 10\ \frac{f}{s^2} \end{array}}$$

EXAMPLE 5: If $\mathbf{r}(t) = \left\langle \frac{t}{2}, t^2 \right\rangle$:

(i) Sketch the curve.

$x = \frac{t}{2} \to t = 2x$

$y = t^2$ $\qquad \boxed{y = 4x^2}$



(ii) Plot the $\underbrace{\text{position}}_{r}$, $\underbrace{\text{tangent}}_{r'}$ and $\underbrace{\text{acceleration}}_{r''}$ vectors at at the point corresponding to $t = 2$.

$r(t) = \left\langle \frac{t}{2}, t^2 \right\rangle \to r(2) = \langle 1, 4 \rangle$ (drawn in red)

$r'(t) = \left\langle \frac{1}{2}, 2t \right\rangle \to r'(2) = \left\langle \frac{1}{2}, 4 \right\rangle$ (drawn in blue)

$r''(t) = \langle 0, 2 \rangle \to r''(2) = \langle 0, 2 \rangle$ (drawn in black)

EXAMPLE 6: Find $f''(x)$ if $f(x) = g(x^3) + (g(x))^3$.

$$f'(x) = \underbrace{g'(x^3)}_{u} \cdot \underbrace{3x^2}_{w} + \underbrace{3(g(x))^2}_{y} \underbrace{g'(x)}_{z}$$

$$f''(x) = \underbrace{g'(x^3)}_{u} \cdot \underbrace{6x}_{w'} + \underbrace{g''(x^3) \cdot 3x^2}_{u'} \cdot \underbrace{3x^2}_{w} + \underbrace{3(g(x))^2}_{y} \underbrace{g''(x)}_{z'} + \underbrace{6(g(x))g'(x)}_{y'} \cdot \underbrace{g'(x)}_{z}$$

EXAMPLE 7: Find y'' by implicit differentiation for the equation $x^2 + \underbrace{6xy^2}_{\text{product rule}} = 8$.

$$2x + (6x)(2yy') + (6)y^2 = 0$$

$$12xyy' = -2x - 6y^2$$

$$y' = \frac{-2x-6y^2}{12xy}$$

$$\boxed{y' = \frac{-x-3y^2}{6xy}} \quad \frac{T}{B}$$

$$y'' = \frac{T'B - TB'}{B^2}$$

$$y'' = \frac{(-1-6yy')(6xy) - (-x-3y^2)(6xy'+6y)}{36x^2y^2}$$