Section 3.9: Slopes and Tangents to Parametric Curves

Derivatives of Parametric Curves: If $x = f(t)$ and $y = g(t)$, then $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$.
This gives us a way to find the slope of the tangent line to the parametric curve at $t = t_0$: $m = \dfrac{dy}{dx}\Big|_{t=t_0}$.

EXAMPLE 1: Find $\dfrac{dy}{dx}$ if $x = (3t-1)^2$ and $y = t\sqrt{t}$. $= t^{\frac{3}{2}}$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{\cancel{3}}{2}t^{\frac{1}{2}}}{2(3t-1)^1(\cancel{3})} \qquad \frac{dy}{dx} = \frac{\sqrt{t}}{4(3t-1)}$$

EXAMPLE 2: If $x = 1-t^3$ and $y = t^2-3t+1$, find an equation of the tangent line corresponding to $t = 2$.

$$m = \frac{dy/dt}{dx/dt}\Bigg|_{t=2}$$

$t=2$: $x = 1-t^3 \rightarrow x = -7$; $y = t^2-3t+1 \rightarrow y = -1$

Point $= (-7,-1)$

$$m = \frac{2t-3}{-3t^2}\Bigg|_{t=2}$$

$$m = -\frac{1}{12}$$

$y - y_1 = m(x - x_1)$

$$\boxed{y+1 = -\frac{1}{12}(x+7)}$$

EXAMPLE 3: If $x = 2t+3$ and $y = t^2+2t$, find the equation of the tangent line at the $\boxed{\text{point } (5,3).}$

$$m = \frac{dy/dt}{dx/dt}\Bigg|_{t=1} \qquad m = \frac{2t+2}{2}\Bigg|_{t=1}$$

$t=1$ yields the point $(5,3)$

$$\boxed{m = 2}$$

$$\boxed{y-3=2(x-5)}$$

EXAMPLE 4: If $x = t^3 - 3t^2$ and $y = t^3 - 3t$, find all points on the curve where the tangent line is vertical or horizontal.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{dy}{dx} = \frac{3t^2-3}{3t^2-6t} \to \frac{dy}{dx} = \frac{3(t+1)(t-1)}{3t(t-2)}$$

① horizontal tangent: $m = 0 \to \frac{dy}{dt} = 0$

$3(t+1)(t-1) = 0$

$t=1 \; \begin{cases} x=-2 \\ y=-2 \end{cases}$

$t=-1 \; \begin{cases} x=-4 \\ y=2 \end{cases}$

$\boxed{\text{horizontal tangents occur at } (-2,-2) + (-4,2)}$

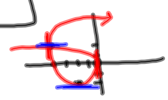
② vertical tangents: undefined slope

$\frac{dx}{dt} = 0 \to 3t(t-2) = 0$

$t=0 \; \begin{cases} x=0 \\ y=0 \end{cases}$

$t=2 \; \begin{cases} x=-4 \\ y=2 \end{cases}$

$\boxed{\text{vertical tangents occur at } (0,0) + (-4,2)}$

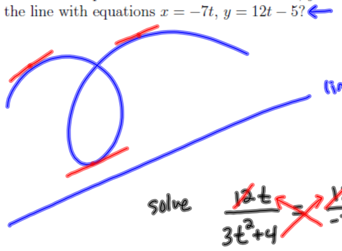


EXAMPLE 5: Show the curve $x = \cos t$ and $y = (\sin t)(\cos t)$ has two tangents at $(0, 0)$. Find the equations of these tangent lines.

$t = \frac{\pi}{2}$ $\quad m = \left.\frac{dy/dt}{dx/dt}\right|_{t=\frac{\pi}{2}} = \left.\frac{\cos^2 t - \sin^2 t}{-\sin t}\right|_{t=\frac{\pi}{2}} = \frac{-1}{-1} = 1$ $\quad m=1$ $\quad (0,0)$ $\quad \boxed{y=x}$

$t = \frac{3\pi}{2}$ $\quad m = \left.\frac{\cos^2 t - \sin^2 t}{-\sin t}\right|_{t=\frac{3\pi}{2}} = \frac{-1}{1} = -1$ $\quad m=-1$ $\quad$ point $(0,0)$ $\quad \boxed{y=-x}$

EXAMPLE 6: At what points on the curve $x = t^3 + 4t$, $y = 6t^2$ is the tangent line parallel to the line with equations $x = -7t$, $y = 12t - 5$?



$m_{curve} = \frac{12t}{3t^2+4}$

$m_{line} = \frac{12}{-7}$

solve $\frac{12t}{3t^2+4} = \frac{12}{-7}$

$-7t = 3t^2 + 4$

$3t^2 + 7t + 4 = 0$

$(3t+4)(t+1) = 0$

$t=-1 \; \begin{cases} x = t^3+4t & x=-5 \\ y = 6t^2 & y=6 \end{cases}$ $\quad$ point $= (-5, 6)$

$t=-\frac{4}{3} \; \begin{cases} x = -\frac{208}{27} \\ y = \frac{32}{3} \end{cases}$ $\quad$ point $= \left(-\frac{208}{27}, \frac{32}{3}\right)$