

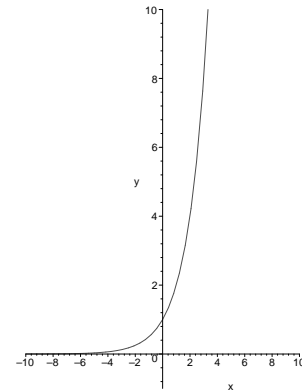
Section 4.1: Exponential Functions and their derivatives

Definition: The function $f(x) = a^x$, $a \neq 1$, $a > 0$, is called an exponential function.

- Case 1: Exponential Growth If $a > 1$, then $f(x) = a^x$ grows exponentially.

Note in particular:

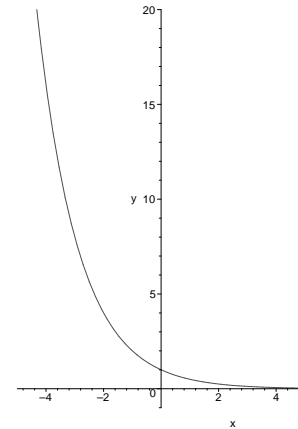
- The domain is $(-\infty, \infty)$.
- The range is $(0, \infty)$.
- $\lim_{x \rightarrow \infty} a^x = \infty$.
- $\lim_{x \rightarrow -\infty} a^x = 0$.



- Case 2: Exponential Decay If $0 < a < 1$, then $f(x) = a^x$ decays exponentially.

Note in particular:

- The domain is $(-\infty, \infty)$.
- The range is $(0, \infty)$.
- $\lim_{x \rightarrow \infty} a^x = 0$.
- $\lim_{x \rightarrow -\infty} a^x = \infty$.



Properties of Exponential Functions:

- $a^{x+y} = a^x a^y$
- $a^{x-y} = \frac{a^x}{a^y}$
- $(a^x)^y = a^{xy}$
- $(ab)^x = a^x b^x$
- $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$

EXAMPLE 1: Sketch the graph of $f(x) = 2^x$ and $g(x) = 3^x$ on the same axis.

EXAMPLE 2: Sketch the graph of $f(x) = \left(\frac{1}{2}\right)^x - 4$ using transformations of graphs.

Definition: We call $f(x) = e^x$ the exponential function, where $e \approx 2.718281828$. One interesting fact about $f(x) = e^x$ is that it is the only exponential function where the slope of the tangent line at $x = 0$ is 1.

EXAMPLE 3: Find the limit:

(a) $\lim_{x \rightarrow \infty} 0.3^{-x}$

(b) $\lim_{x \rightarrow -\infty} 0.3^{-x}$

(c) $\lim_{x \rightarrow 2^+} \left(\frac{1}{4}\right)^{\frac{x}{2-x}}$

(d) $\lim_{x \rightarrow 2^-} \left(\frac{1}{4}\right)^{\frac{x}{2-x}}$

$$(e) \lim_{x \rightarrow 1^-} e^{\frac{2}{x-1}}$$

$$(f) \lim_{x \rightarrow 1^+} e^{\frac{2}{x-1}}$$

$$(g) \lim_{x \rightarrow \infty} \frac{e^x - e^{-3x}}{e^{3x} + e^{-3x}}$$

$$(h) \lim_{x \rightarrow -\infty} \frac{2^{-x} + 2^x}{4^{-x} + 3^x}$$

Derivatives of Exponential Functions

(i) $\frac{d}{dx}e^x = e^x$

(ii) $\frac{d}{dx}e^{g(x)} = g'(x)e^{g(x)}$

EXAMPLE 4: Find the derivative.

(a) $y = \sqrt{e^x + x} + \frac{1}{e} + x^e$

(b) $f(x) = e^{-5x} \cos(3x)$

(c) $f(x) = e^{x \sin x}$

EXAMPLE 5: Find the equation of the tangent line to the graph of $2e^{xy} = x + y$ at the point $(0, 2)$.

EXAMPLE 6: For what value(s) of r does $y = e^{rx}$ satisfy $y + y' = y''$?

EXAMPLE 7: Find the equation of the tangent line to the parametric curve $x = e^{-t}$, $y = te^{2t}$ at $t = 0$.

EXAMPLE 8: Find the derivative of $f(x) = g(e^x) + e^{g(\sin x)}$.