Section 4.1: Exponential Functions and their derivatives

Definition: The function $f(x) = a^x$, $a \neq 1$, $a > 0$, is called an exponential function.

- Case 1: Exponential Growth If $a > 1$, then $f(x) = a^x$ grows exponentially.

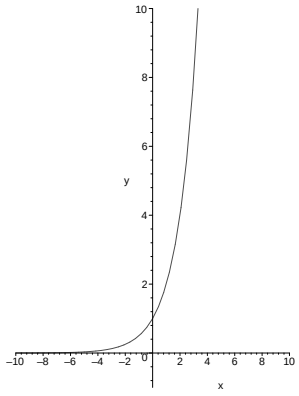
  Note in particular:

  (i) The domain is $(-\infty, \infty)$.

  (ii) The range is $(0, \infty)$.

  (iii) $\lim_{x \to \infty} a^x = \infty$.

  (iv) $\lim_{x \to -\infty} a^x = 0$.



- Case 2: Exponential Decay If $0 < a < 1$, then $f(x) = a^x$ decays exponentially.

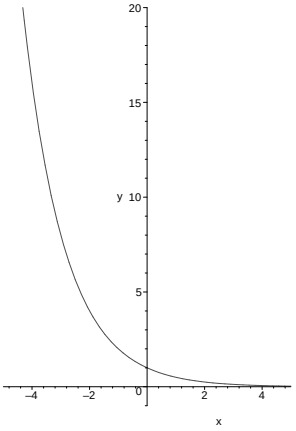
  Note in particular:

  (i) The domain is $(-\infty, \infty)$.

  (ii) The range is $(0, \infty)$.

  (iii) $\lim_{x \to \infty} a^x = 0$.

  (iv) $\lim_{x \to -\infty} a^x = \infty$.



EXAMPLE 1: Sketch the graph of $f(x) = 2^x$ and $g(x) = 3^x$ on the same axis. Using the definition of the derivative, estimate $f'(0)$ and $g'(0)$ by making a table of values.

Definition: e is the number such that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$.

EXAMPLE 2: Sketch the graph of $f(x) = \left(\frac{1}{2}\right)^x - 4$ using transformations of graphs.

EXAMPLE 3: Find the limit:

(a) $\lim_{x \to -\infty} \left(\frac{\pi}{4}\right)^x$

(b) $\lim_{x \to \infty} \left(\frac{2\pi}{7}\right)^x$

(c) $\lim_{x \to 2^+} \left(\frac{1}{4}\right)^{\frac{x}{2-x}}$

(d) $\lim_{x \to 1^-} e^{\frac{2}{x-1}}$

(e) $\lim_{x \to 1^+} e^{\frac{2}{x-1}}$

(f) $\lim_{x\to\infty} \dfrac{e^x - e^{-3x}}{e^{3x} + e^{-3x}}$

(g) $\lim_{x\to-\infty} \dfrac{2^{-x} + 2^x}{4^{-x} + 3^x}$

Properties of Exponential Functions:

(i) $a^{x+y} = a^x a^y$

(ii) $a^{x-y} = \dfrac{a^x}{a^y}$

(iii) $(a^x)^y = a^{xy}$

(iv) $(ab)^x = a^x b^x$

Derivatives of Exponential Functions

(i) $\dfrac{d}{dx}e^x = e^x$

(ii) $\dfrac{d}{dx}e^{g(x)} = g'(x)e^{g(x)}$

EXAMPLE 4: Find the derivative.

(a) $y = \sqrt{e^x + x}$

(b) $f(x) = e^{-5x}\cos(3x)$

(c) $f(x) = e^{x\sin x}$

EXAMPLE 5: Find the equation of the tangent line to the graph of $2e^{xy} = x + y$ at the point $(0, 2)$.

EXAMPLE 6: For what value(s) of r does $y = e^{rx}$ satisfy $y + y' = y''$?

EXAMPLE 7: Find the equation of the tangent line to the parametric curve $x = e^{-t}$, $y = te^{2t}$ at $t = 0$.

EXAMPLE 8: Find the derivative of $f(x) = g(e^x) + e^{g(\sin x)}$.