

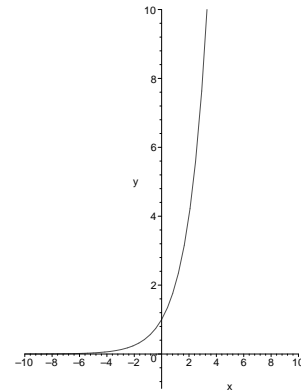
## Section 4.1: Exponential Functions and their derivatives

**Definition:** The function  $f(x) = a^x$ ,  $a \neq 1$ ,  $a > 0$ , is called an exponential function.

- Case 1: Exponential Growth If  $a > 1$ , then  $f(x) = a^x$  grows exponentially.

Note in particular:

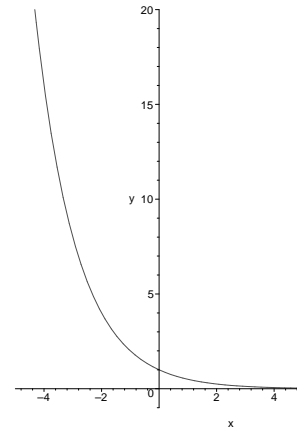
- The domain is  $(-\infty, \infty)$ .
- The range is  $(0, \infty)$ .
- $\lim_{x \rightarrow \infty} a^x = \infty$ .
- $\lim_{x \rightarrow -\infty} a^x = 0$ .



- Case 2: Exponential Decay If  $0 < a < 1$ , then  $f(x) = a^x$  decays exponentially.

Note in particular:

- The domain is  $(-\infty, \infty)$ .
- The range is  $(0, \infty)$ .
- $\lim_{x \rightarrow \infty} a^x = 0$ .
- $\lim_{x \rightarrow -\infty} a^x = \infty$ .



**Properties of Exponential Functions:**

- $a^{x+y} = a^x a^y$
- $a^{x-y} = \frac{a^x}{a^y}$
- $(a^x)^y = a^{xy}$
- $(ab)^x = a^x b^x$
- $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$

*EXAMPLE 1:* Sketch the graph of  $f(x) = 2^x$  and  $g(x) = 3^x$  on the same axis.

**Definition:** We call  $f(x) = e^x$  the exponential function, where  $e \approx 2.718281828$ . One interesting fact about  $f(x) = e^x$  is that it is the only exponential function where the slope of the tangent line at  $x = 0$  is 1.

*EXAMPLE 2:* Find the limit:

(a)  $\lim_{x \rightarrow \infty} 0.3^{-x}$

(b)  $\lim_{x \rightarrow -\infty} \left(\frac{e}{\pi}\right)^x$

$$(c) \lim_{x \rightarrow 2^+} \left(\frac{1}{4}\right)^{\frac{x}{2-x}}$$

$$(d) \lim_{x \rightarrow 2^-} \left(\frac{1}{4}\right)^{\frac{x}{2-x}}$$

$$(d) \lim_{x \rightarrow -\infty} \frac{4}{1 + e^x}$$

$$(e) \lim_{x \rightarrow \infty} \frac{e^x - e^{-3x}}{e^{3x} + e^{-3x}}$$

**Derivatives of Exponential Functions**

(i)  $\frac{d}{dx}e^x = e^x$

(ii)  $\frac{d}{dx}e^{g(x)} = g'(x)e^{g(x)}$

*EXAMPLE 3:* Find the derivative.

(a)  $y = \sqrt{e^x + x} + \frac{1}{e} + x^e$

(b)  $f(x) = e^{-5x} \cos(3x)$

(c)  $f(x) = e^{x \sin x}$

*EXAMPLE 4:* Find the equation of the tangent line to the graph of  $2e^{xy} = x + y$  at the point  $(0, 2)$ .

*EXAMPLE 5:* Find the 800th derivative of  $f(x) = xe^x$

*EXAMPLE 6:* Find the equation of the tangent line to the parametric curve  $x = e^{-t}$ ,  $y = te^{2t}$  at  $t = 0$ .

*EXAMPLE 7:* Find the derivative of  $f(x) = g(e^x) + e^{g(\sin x)}$ .