Section 4.5: Exponential Growth and Decay

Definition: If $y(t)$ is the value of a quantity at time t and if the rate of change of y with respect to t is proportional to its size $y(t)$ at any time, [That is $\underbrace{\frac{dy}{dt} = ky}$], then the quantity $y(t)$ at time t is given by

$$\boxed{y(t) = y_0 e^{kt}} \leftarrow \text{result}$$

where y_0 is the initial quantity and k is a constant. Given information, your primary goal is to find k.

EXAMPLE 1: A bacteria culture starts with 4000 bacteria and the population triples every half-hour. → $\frac{1}{2}$ hour

(i) Find an expression for the number of bacteria after t hours.

(ii) Find the number of bacteria after 20 minutes.

$y(t) = \cancel{y_0}^{4000} e^{kt}$ given $y(\frac{1}{2}) = 3(4000)$

$3(4\cancel{000}) = 4\cancel{000}\, e^{k(\frac{1}{2})}$

$3 = e^{\frac{1}{2}k} \rightarrow \ln 3 = \frac{1}{2}k \rightarrow k = 2\ln 3$

$\boxed{k = \ln 9}$

$y(t) = 4000 e^{(\ln 9)t}$

Fact: $e^{\ln x} = x$

$= 4000 e^{t \ln 9}$

$\ln a^x = x \ln a$

$= 4000 e^{\ln 9^t}$

(i) $\boxed{y(t) = (4000)9^t}$

(ii) 20 minutes $= \frac{1}{3}$ hour

$y(\frac{1}{3}) = (4000)9^{\frac{1}{3}}$ bacteria

≈ 8320 bacteria

EXAMPLE 2: Def: The <u>half-life</u> of a substance is the amount of time it takes for half of the substance to disintegrate. Polonium-210 has a half-life of 140 days.

(i) If a sample has a mass of 200 mg, find a formula for the mass that remains after t days.

(ii) When will the mass be reduced to 10 mg?

$y(t) = y_0 e^{kt}$

$y(t) = 200 e^{kt}$

$y(140) = \frac{1}{2}(200)$

$100 = 200 e^{k(140)}$

$\frac{1}{2} = e^{140k}$

$\ln \frac{1}{2} = 140k$

$k = \frac{1}{140} \ln \frac{1}{2}$

(i) $y(t) = 200 e^{\frac{t}{140}\ln\frac{1}{2}}$

$= 200 e^{\ln\left(\frac{1}{2}\right)^{\frac{t}{140}}}$

$\boxed{y(t) = 200\left(\frac{1}{2}\right)^{\frac{t}{140}}}$

(ii) Solve $y(t) = 10$.

$10 = 200\left(\frac{1}{2}\right)^{t/140}$

$\frac{1}{20} = \left(\frac{1}{2}\right)^{t/140}$

$\ln \frac{1}{20} = \ln\left(\frac{1}{2}\right)^{t/140}$

$\ln \frac{1}{20} = \frac{t}{140} \ln\left(\frac{1}{2}\right)$

$t = 140 \frac{\ln \frac{1}{20}}{\ln \frac{1}{2}}$ days ≈ 605 days

EXAMPLE 3: After 3 days a sample of radon-222 decayed to 58% of its original amount. What is the half-life of radon-222? How long will it take the sample to decay to 10% of its original amount?

$y(t) = y_0 e^{kt}$

given $y(3) = (.58) y_0$

$.58 \cancel{y_0} = \cancel{y_0} e^{k(3)}$

$.58 = e^{3k}$

$\ln(.58) = 3k$

$k = \frac{1}{3}\ln(.58)$

$y(t) = y_0 e^{\frac{t}{3}\ln(.58)}$

$y(t) = y_0 (.58)^{\frac{t}{3}}$

Find the half-life

Solve $y(t) = \frac{1}{2} y_0$

$\frac{1}{2} y_0 = y_0 (.58)^{\frac{t}{3}}$

$\frac{1}{2} = (.58)^{t/3}$

$\ln \frac{1}{2} = \frac{t}{3}\ln(.58)$

$t = 3 \frac{\ln \frac{1}{2}}{\ln(.58)} \approx 3.82$ days

How long will it take for sample to be reduced to 10% of its original amount?

Solve $y(t) = .10 y_0 \rightarrow .10 y_0 = y_0 (.58)^{\frac{t}{3}}$

$\vdots$

___ days.

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EXAMPLE 4: A curve passes through the point $(0, 7)$ and has the property that the slope of the curve at every point p is half the y-coordinate of p. Find the equation of the curve.

Recall: $\frac{dy}{dt} = ky \rightarrow y(t) = y_0 e^{kt}$

let $y(x)$ be the equation of the curve.

$\boxed{y(0) = 7}$ given: $\frac{dy}{dx} = \frac{1}{2}y \rightarrow y(x) = y_0 e^{\frac{1}{2}x}$

$y(x) = 7e^{\frac{1}{2}x}$

check: $y'(x) = \frac{7}{2}e^{\frac{1}{2}x} = \frac{1}{2} \cdot 7e^{\frac{1}{2}x}$

EXAMPLE 5: The rate of change of atmospheric pressure P with respect to altitude h is proportional to P, provided that the temperature is constant. At a specific temperature the pressure is 101 kPa at sea level and 86.9 kPa at h = 1,000 m. What is the pressure at an altitude of 3500 m?

$\frac{dP}{dh} = kP \rightarrow P(h) = P_0 e^{kh}$ $\quad P_0 = 101$

$P(h) = 101e^{kh}$ $\quad P(1000) = 86.9$

$86.9 = 101e^{k(1000)}$

$\frac{86.9}{101} = e^{1000k}$

$\ln\frac{86.9}{101} = 1000k$

$k = \frac{1}{1000}\ln\frac{86.9}{101}$

$P(h) = 101e^{\frac{h}{1000}\ln\frac{86.9}{101}}$

$P(h) = 101\left(\frac{86.9}{101}\right)^{\frac{h}{1000}}$

$P(3500) = 101\left(\frac{86.9}{101}\right)^{\frac{3500}{1000}}$

≈ 59.67 kPa

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Compound Interest: If A_0 dollars is invested at $r\%$ compounded n times a year, then the amount in the account after t years is given by $A = A_0(1 + r/n)^{nt}$.

EXAMPLE 6: If \$4000 is invested at 8% compounded monthly, how much money is in the account at the end of 6 years?

$$A = \$4000\left(1 + \frac{.08}{12}\right)^{12t}$$

$$A(6) = \$4000\left(1 + \frac{.08}{12}\right)^{72} \approx \$6454.00$$

Continuous Compound Interest: If P dollars is invested at $r\%$ compounded continuously, then the amount in the account after t years is given by $A = Pe^{rt}$.

EXAMPLE 7: How much money should be invested now at 6% compounded continuously in order to have \$30,000 18 years from now?

$$30{,}000 = Pe^{(0.06)(18)}$$

$$P = \frac{30{,}000}{e^{(.06)(18)}} \approx \$10{,}187.87$$

Definition: The rate of cooling of an object is proportional to the temperature difference between the object and the temperature of the object's surroundings. If $y(t)$ is the temperature of the object at time t, then $\frac{dy}{dt} = k(y - T)$, where y is the temperature of the object at time t and T is the room temperature (the temperature of the room in which the object is cooling). The solution of this equation, which gives the temperature of the object at time t, is $y(t) = (y_0 - T)e^{kt} + T$, where y_0 is the initial temperature of the object.

y= temp of object at time t

T= temp of cooling room

$\frac{dy}{dt} = k(y - T)$

$y(t) = (y_0 - T)e^{kt} + T$

EXAMPLE 8: A thermometer is taken from a room where the temperature is 20°C to the outdoors, where the temperature is 5° C. After one minute, the temperature reads 12°C. Use Newton's Law of Cooling to answer the following questions.

a.) What will the reading of the thermometer be after 2 minutes?

b.) When will the thermemeter read 6°C?

$y_0 = 20°$

$T = 5°$

$y(1) = 12°$

$y(t) = 15e^{kt} + 5$

$12 = 15e^{k(1)} + 5$

$7 = 15e^{k} \rightarrow \frac{7}{15} = e^{k} \rightarrow \boxed{k = \ln\frac{7}{15}}$

$y(t) = 15e^{t\ln\frac{7}{15}} + 5 \rightarrow y(t) = 15\left(\frac{7}{15}\right)^{t} + 5$

(i) $y(2) = 15\left(\frac{7}{15}\right)^{2} + 5 \approx 8.3°$

(ii) solve $y(t) = 6$

$6 = 15\left(\frac{7}{15}\right)^{t} + 5$

$1 = 15\left(\frac{7}{15}\right)^{t}$

$\frac{1}{15} = \left(\frac{7}{15}\right)^{t} \rightarrow \ln\frac{1}{15} = t\ln\frac{7}{15}$

$t = \dfrac{\ln\frac{1}{15}}{\ln\frac{7}{15}}$ min

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