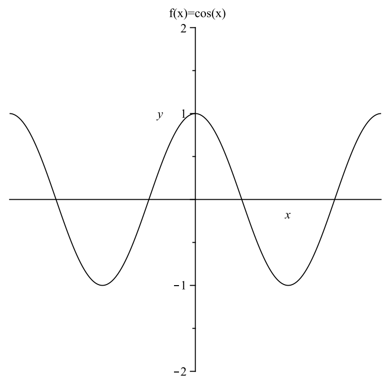
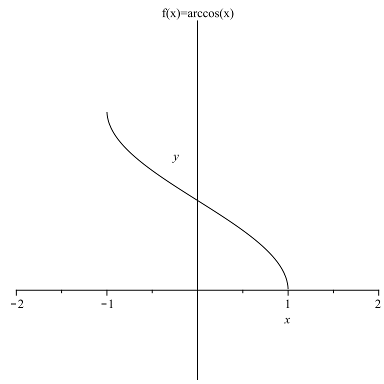
Section 4.6: Inverse Trigonometric Functions

I. INVERSE COSINE: If $0 \leq x \leq \pi$, then $f(x) = \cos x$ is one-to-one, thus the inverse exists, denoted by $\cos^{-1}(x)$ or $\arccos x$. Additionally, the domain of $\arccos x$ = range of $\cos x = [-1, 1]$ and range of $\arccos x$ = domain of $\cos x = [0, \pi]$. Note: $\arccos(x)$ is the **angle** in $[0, \pi]$ whose cosine is x.





Cancellation Equations: Recall $f^{-1}(f(x)) = x$ for x in the domain of f, and $f(f^{-1}(x)) = x$ for x in the domain of f^{-1}. This yields the following cancellation equations:

- $\arccos(\cos x) = x$ if $0 \leq x \leq \pi$
- $\cos(\arccos x) = x$ if $-1 \leq x \leq 1$.

Example 1: Compute the following.

(i) $\arccos(0)$

(ii) $\cos^{-1}(1)$

(iii) $\arccos(-1)$

(iv) $\arccos \dfrac{1}{2}$

(v) $\cos^{-1}\dfrac{-\sqrt{3}}{2}$

(vi) $\sin\left(2\arccos(-\dfrac{4}{5})\right)$

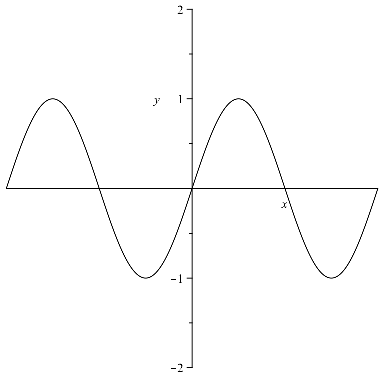
(vii) $\arccos\left((\cos(\dfrac{\pi}{6})\right)$

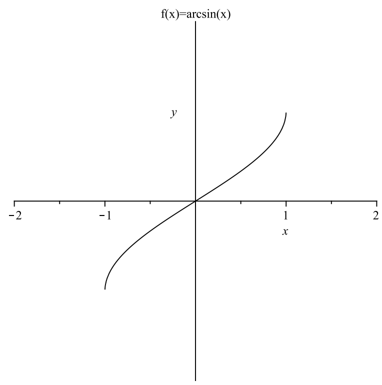
(viii) $\arccos\left((\cos(\dfrac{7\pi}{6})\right)$

(ix) $\arccos\left((\cos(-\dfrac{\pi}{3})\right)$

(x) $\cos\left((\cos^{-1}(2)\right)$

II. INVERSE SINE: If $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, then $f(x) = \sin x$ is one-to-one, thus the inverse exists, denoted by $\sin^{-1}(x)$ or $\arcsin x$. Additionally, the domain of $\arcsin x =$ range of $\sin x = [-1, 1]$ and range of $\arcsin x =$ domain of $\sin x = [-\frac{\pi}{2}, \frac{\pi}{2}]$.
Note: $\arcsin(x)$ is the **angle** in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ whose sine is x.





Cancellation Equations: Recall $f^{-1}(f(x)) = x$ for x in the domain of f, and $f(f^{-1}(x)) = x$ for x in the domain of f^{-1}. This yields the following cancellation equations:

- $\arcsin(\sin x) = x$ if $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$
- $\sin(\arcsin x) = x$ if $-1 \le x \le 1$.

Example 3: Compute the following.

(i) $\arcsin(0)$

(ii) $\sin^{-1}(1)$

(iii) $\arcsin(-1)$

(iv) $\arcsin \frac{1}{2}$

(v) $\sin^{-1} \dfrac{-\sqrt{3}}{2}$

(vi) $\tan\left(\arcsin(\dfrac{2}{3})\right)$

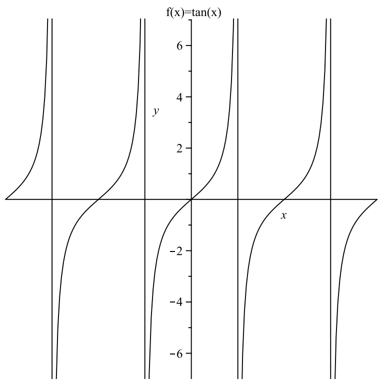
(vii) $\sin\left(\arcsin(\dfrac{3}{10})\right)$

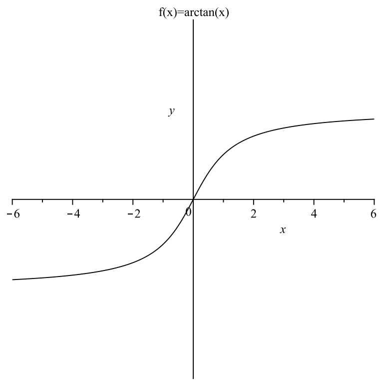
(viii) $\arcsin\left((\sin(\dfrac{5\pi}{4})\right)$

(ix) $\arcsin\left((\sin(-\dfrac{\pi}{6})\right)$

(x) $\arcsin\left(\sin \dfrac{\pi}{120}\right)$

III. INVERSE TANGENT: If $-\frac{\pi}{2} < x < \frac{\pi}{2}$, then $f(x) = \tan x$ is one-to-one, thus the inverse exists, denoted by $\tan^{-1}(x)$ or $\arctan x$. Additionally, the domain of $\arctan x$ = range of $\tan x = (-\infty, \infty)$ and range of $\arctan x$ = domain of $\tan x = (-\frac{\pi}{2}, \frac{\pi}{2})$. Note: $\arctan(x)$ is the **angle** in $(-\frac{\pi}{2}, \frac{\pi}{2})$ whose tangent is x.





Cancellation Equations: Recall $f^{-1}(f(x)) = x$ for x in the domain of f, and $f(f^{-1}(x)) = x$ for x in the domain of f^{-1}. This yields the following cancellation equations:

- $\arctan(\tan x) = x$ if $-\frac{\pi}{2} < x < \frac{\pi}{2}$
- $\tan(\arctan x) = x$ for all x.

Example 5: Compute the following.

(i) $\arctan(0)$

(ii) $\tan^{-1}(1)$

(iii) $\arctan(-1)$

(iv) $\arctan(-\sqrt{3})$

(v) $\tan(\arcsin x)$

(vi) $\arctan\left(\tan(\frac{5\pi}{3})\right)$

(vii) $\lim_{x\to\infty} \arctan x$

(viii) $\lim_{x\to-\infty} \arctan x$

Derivatives of Inverse Trigonometric Functions:

A.) $\dfrac{d}{dx}\arcsin x = \dfrac{1}{\sqrt{1-x^2}}$.

B.) $\dfrac{d}{dx}\arccos x = -\dfrac{1}{\sqrt{1-x^2}}$.

C.) $\dfrac{d}{dx}\arctan x = \dfrac{1}{1+x^2}$.

Example 7: Prove formula A.

Example 8: Find the derivative of $f(x) = \arccos(2x - 1)$.

Example 9: Find the derivative of $f(x) = \tan^{-1}(\arcsin x)$.

Example 10: What is the domain of $\arcsin(3x + 1)$? Of $\arctan(3x + 1)$?

The Unit Circle

Points of Special Interest on the Unit Circle:

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