Section 4.8: L'Hospital's Rule

Indeterminate form: If $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$, then we say the limit is in indeterminate form.

L'Hospital's Rule: If $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$, then $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$.

Some common misconceptions: If If $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{0}{\infty}$ or $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\infty}{0}$, the limit is NOT indeterminate! For example,

(i) $\lim\limits_{x\to 0^+}\dfrac{\ln x}{\sqrt{x}}$

(ii) $\lim\limits_{x\to 0^+}\dfrac{x}{\ln x}$

Example 1: Find the following limits, if they exist. If the limit does not exist, explain why.

(i) $\lim\limits_{x\to 1}\dfrac{\ln x}{x-1}$

(ii) $\lim\limits_{x\to 0}\dfrac{\sin x - x}{x^3}$

(iii) $\lim\limits_{x\to 0}\dfrac{\sin mx}{\sin nx}$

(iv) $\lim\limits_{x\to\infty} \dfrac{(\ln x)^2}{x}$

Indeterminate Products: If $\lim\limits_{x\to a} f(x)g(x) = 0\cdot\infty$, this limit is an indeterminate product. Why do we call the product indeterminate?

$\lim\limits_{x\to\infty} \dfrac{1}{x^2}\cdot x$ $\qquad$ $\lim\limits_{x\to\infty} \dfrac{1}{x}\cdot x^2$ $\qquad$ $\lim\limits_{x\to\infty} \dfrac{1}{x^2}\cdot 6x^2$

All three of these limits are of the form $0\cdot\infty$, yet they all have different limits. The goal is ty try to manipulate the product get the limit in the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$, then use L'Hospital's rule.

Example 2: Find the following limits, if they exist. If the limit does not exist, explain why.

(i) $\lim\limits_{x\to 0^+} x^3\ln x$

(ii) $\lim\limits_{x\to 1^+} (x-1)\tan(\pi x/2)$

Indeterminate Powers: If $\lim_{x\to a} f(x)^{g(x)}$ is of the form 0^0, ∞^0 or 1^∞, then the limit is an indeterminate power. To solve such a limit, take the natural logarithm, which converts the indeterminate power into an indeterminate product.

Example 3: Find the following limits, if they exist. If the limit does not exist, explain why.

(i) $\lim_{x\to\infty} x^{\frac{3}{x}}$

(ii) $\lim_{x\to\infty} \left(\frac{2x-3}{2x+5}\right)^{2x+1}$

Indeterminate difference: If $\lim_{x\to a}(f(x) - g(x)) = \infty - \infty$, this limit is an indeterminate difference.

Example 4: Find $\lim_{x\to 1} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$