Section 5.3: Derivatives and Shapes of Curves

Mean Value Theorem If $f(x)$ is continuous on the interval $[a, b]$ and differentiable on the interval (a, b), then there exists a number c, where $a < c < b$, so that $f'(c) = \frac{f(b) - f(a)}{b - a}$. Graphically, this means the tangent line to the graph of $f(x)$ at $x = c$ is parallel to the secant line joining the points $(a, f(a))$ and $(b, f(b))$.
Illustration:

EXAMPLE 1: Given $f(x) = 4 - x^2$, show $f(x)$ satisfies the mean value theorem on the interval $[1, 2]$ and find all c that satisfy the conclusion of the mean value theorem.

EXAMPLE 2: Suppose $1 \leq f'(x) \leq 4$ for all x in the interval $[2, 5]$.
Prove $3 \leq f(5) - f(2) \leq 12$.

First derivative test for increasing/decreasing and local extrema.

- If $f' > 0$ on an interval I, then f is increasing on I.
- If $f' < 0$ on an interval I, then f is decreasing on I.
- If f' goes from positive to negative at $x = a$, and $x = a$ is in the domain of f, then f has a local maximum at $x = a$.
- If f' goes from negative to positive at $x = a$, and $x = a$ is in the domain of f, then f has a local minimum at $x = a$.

EXAMPLE 3: Find all intervals of increase and decrease and identify all local extrema:

(a) $f(x) = x^4 + 4x^3 + 3$

(b) $f(x) = x\sqrt{x+1}$

(c) $f(x) = xe^{2x}$

Second derivative test for concavity and inflection points.

- If $f'' > 0$ on an interval I, then f' is increasing, hence f is concave up on I.
- If $f'' < 0$ on an interval I, then f' is decreasing, hence f is concave down on I.
- If f changes concavity at $x = a$, and $x = a$ is in the domain of f, then $x = a$ is an inflection point of f.

EXAMPLE 4: Find intervals of concavity and inflection points of we are given that $f'(x) = 4x^3 - 6x + 6$.

EXAMPLE 5: Sketch the graph of $f(x) = x^4 - 6x^2$ by locating intervals of increase/decrease, local extrema, concavity and inflection points.

EXAMPLE 6: Sketch the graph of $f(x) = \frac{x}{(x-1)^2}$ by locating intervals of increase/decrease, local extrema, concavity and inflection points.

Second derivative test for local extrema. If $x = c$ is a critical number for $f(x)$, then:

- If $f''(c) > 0$, then f is concave up, therefore $f(x)$ has a local minimum at $x = c$.
- If $f''(c) < 0$, then f is concave down, therefore $f(x)$ has a local maximum at $x = c$.
- If $f''(c) = 0$ or does not exist, then the test fails, therefore use the first derivative test to find the local extrema.

EXAMPLE 6: Use the second derivative test to find the local extrema for $f(x) = x^3 - 3x - 1$.

EXAMPLE 7: Find a cubic function $f(x) = ax^3 + bx^2 + cx + d$ that has a local maximum value of 3 at -2 and a local minimum value of 0 at 1.