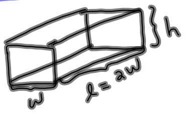
EXAMPLE 1: A company wants to manufacture a box with a volume of 36 cubic feet. The box has no top, and the length is twice the width. Find the dimensions of the box that minimizes the amount of material used.



constraint - part of the problem that is fixed.

$\boxed{V = 36\ ft^3} \rightarrow 2w^2h = 36 \rightarrow h = \frac{18}{w^2}$

what do we want to do?

minimize $SA = \underbrace{2w^2}_{\text{bottom}} + \underbrace{2 \cdot wh}_{\text{two ends}} + \underbrace{2 \cdot 2wh}_{\text{two sides}}$

$\boxed{SA = 2w^2 + 6wh}$

$SA = 2w^2 + 6w\left(\frac{18}{w^2}\right)$

$\boxed{SA = 2w^2 + \frac{108}{w}} \rightarrow SA' = 4w - \frac{108}{w^2}$

$SA' = \frac{4w^3 - 108}{w^2}$

cn: $SA' = 0 \rightarrow 4w^3 - 108 = 0$

$4w^3 = 108$

$w^3 = 27$

$\boxed{w = 3}$

show $w = 3$ yields a minimum:

$\searrow \nearrow$

$- \quad +$

$2 \quad 3 \quad 4$

proves $w = 3$ gives a minimum.

$w = 3$ ft

$\ell = 2w = 6$ ft

$h = \frac{18}{w^2} = \frac{18}{9} = 2$ feet

$\boxed{3 \times 6 \times 2\ ft^3}$

EXAMPLE 2: If ~~10,800~~ square centimeters of material is available to make a box with a ~~square base~~ and an ~~open top~~, find the largest possible volume of the box.

↳ no lid

constraint: $SA = 10,800\ cm^2$

maximize $V = x^2h \rightarrow V = x^{\cancel{2}}\left(\frac{10,800 - x^2}{4\cancel{x}}\right)$

$SA = x^2 + 4xh = 10,800$

$4xh = 10,800 - x^2$

$h = \frac{10,800 - x^2}{4x}$

$V = \frac{x}{4}(10,800 - x^2)$

$V = 2700x - \frac{x^3}{4}$

$* V' = 2700 - \frac{3x^2}{4} *$

$V' = 0$

$2700 - \frac{3x^2}{4} = 0$

$2700 = \frac{3x^2}{4}$

$(2700)\left(\frac{4}{3}\right) = x^2$

$3600 = x^2$

$\boxed{x = 60}$

To prove $x = 60$ yields a max, use second derivative test: $V'' = \frac{-6x}{4} < 0$

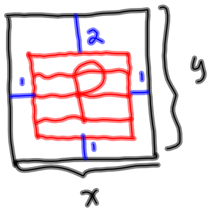
V is concave down

$\therefore$ $x = 60$ yields a max

$V = 2700(60) - \frac{60^3}{4}$

$V = 108,000\ cm^3$

EXAMPLE 3: A poster is to have an area of 180 square inches with 1-inch margins at the bottom and sides and a 2-inch margin at the top. What dimensions will give the largest printed area?



constraint: $xy = 180 \rightarrow y = \frac{180}{x}$

maximize $P = (y-3)(x-2)$

$$P = \left(\frac{180}{x} - 3\right)(x-2)$$

$$P = 180 - \frac{360}{x} - 3x + 6$$

show max using second deriv test:

$$*P' = \frac{360}{x^2} - 3*$$

$$P'' = -\frac{720}{x^3} < 0$$

P concave down

local max at $x = \sqrt{120}$

$$P' = 0 \rightarrow \frac{360}{x^2} - 3 = 0$$

$$\frac{360}{x^2} = 3$$

$$360 = 3x^2$$

$$120 = x^2$$

$$x = \sqrt{120}$$

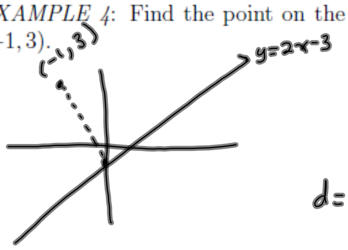
poster dimensions are

$x = \sqrt{120}$, $y = \frac{180}{x}$

$$y = \frac{180}{\sqrt{120}}$$

$$\boxed{\text{Dimensions: } \sqrt{120} \times \frac{180}{\sqrt{120}} \text{ in}^2}$$

EXAMPLE 4: Find the point on the line $y = 2x - 3$ that is closest to the point $(-1, 3)$.



$P(-1,3)$

$Q(x, 2x-3)$

$$d = \sqrt{(2x-6)^2 + (x+1)^2}$$

$$d' = \frac{1}{2}\left((2x-6)^2 + (x+1)^2\right)^{-\frac{1}{2}}\left(2(2x-6)(2) + 2(x+1)\right)$$

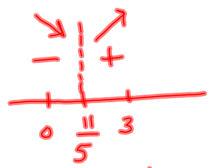
$$= \frac{\cancel{2}(2x-6)(2) + \cancel{2}(x+1)}{\cancel{2}\sqrt{(2x-6)^2 + (x+1)^2}}$$

cn: $(2x-6)(2) + x + 1 = 0$

$$4x - 12 + x + 1 = 0$$

$$5x - 11 = 0$$

$$\boxed{x = \frac{11}{5}}$$



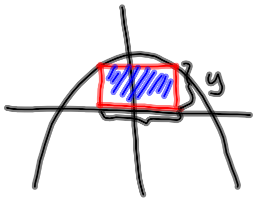
min @ $x = \frac{11}{5}$

$$y = 2x - 3$$

$$y = \frac{22}{5} - 3 = \frac{7}{5}$$

$$\boxed{\text{point} = \left(\frac{11}{5}, \frac{7}{5}\right)}$$

EXAMPLE 5: Find the dimensions of the rectangle of largest area that has its base on the x axis and its other two vertices above the x axis lying on the parabola $\boxed{y = 8 - x^2.}$ ← constraint



$A = 2xy$

$A = 2x(8-x^2)$

$A = 16x - 2x^3$

$* A' = 16 - 6x^2 *$

$A' = 0 \rightarrow 16 - 6x^2 = 0$

$16 = 6x^2$

$\frac{8}{3} = x^2$

$x = \sqrt{\frac{8}{3}}$

use second deriv test:

$A'' = -12x < 0$

concave down proves max.

dimensions:

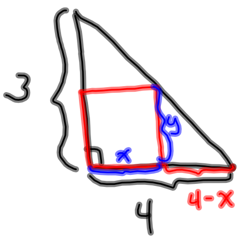
base $= 2x = 2\sqrt{\frac{8}{3}}$

height $= y = 8 - x^2$

$= 8 - \frac{8}{3} = \frac{16}{3}$

$$\boxed{2\sqrt{\frac{8}{3}} \times \frac{16}{3}}$$

EXAMPLE 6: Find the area of the largest rectangle that can be inscribed in a right triangle with legs of lengths 3 cm and 4 cm if two sides of the rectangle lie along the legs.



$\frac{y}{4-x} = \frac{3}{4}$

$y = \frac{3}{4}(4-x)$

maximize $A = xy$

$A = x \cdot \frac{3}{4}(4-x)$

$A = 3x - \frac{3}{4}x^2$

$* A' = 3 - \frac{6}{4}x *$

$A' = 0 \rightarrow 0 = 3 - \frac{3}{2}x$

$3 = \frac{3}{2}x \rightarrow x = 3 \cdot \frac{2}{3} = \boxed{2}$

verify max: $A'' = -\frac{6}{4} < 0$

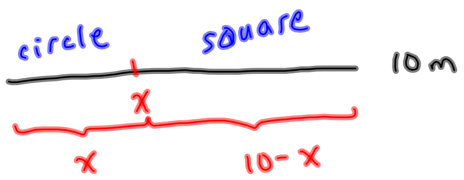
concave down maximum!

$A = 3(2) - \frac{3}{4}(4)$

$A = 6 - 3$

$\boxed{A = 3 \text{ cm}^2}$

EXAMPLE 7: A piece of wire 10 m long is cut into two pieces. One piece is bent into a square and the other is bent into a circle. How should the wire be cut so that the total enclosed area is a maximum? A minimum?



x = # of m bent into a circle.

$10 - x$ = # of m bent into a square.

$0 \le x \le 10$

Total Area is $A = A_o + A_\square$

$A_o = \pi r^2$

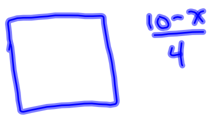
$C = 2\pi r = x$

$r = \frac{x}{2\pi}$

$A_o = \pi\left(\frac{x}{2\pi}\right)^2$

$A_o = \pi\left(\frac{x^2}{4\pi^2}\right)$

$\boxed{A_o = \frac{x^2}{4\pi}}$



$A_\square = s^2$

$A_\square = \left(\frac{10-x}{4}\right)^2$

$\boxed{A_\square = \frac{1}{16}(10-x)^2}$

$\boxed{A = \frac{x^2}{4\pi} + \frac{1}{16}(10-x)^2}$

$A' = \frac{2x}{4\pi} + \frac{1}{8}(10-x)(-1)$

$A' = \frac{x}{2\pi} - \frac{10}{8} + \frac{1}{8}x$

$A' = \frac{1}{2\pi}x + \frac{1}{8}x - \frac{5}{4}$

$A' = x\left(\frac{1}{2\pi} + \frac{1}{8}\right) - \frac{5}{4}$

$A' = 0$

$x\left(\frac{1}{2\pi} + \frac{1}{8}\right) = \frac{5}{4}$

$x\left(\frac{4+\pi}{8\pi}\right) = \frac{5}{4}$

$x = \frac{5}{4} \cdot \frac{8\pi}{4+\pi}$

$\boxed{x = \frac{10\pi}{4+\pi}}$

$A(0) = \frac{100}{16} = 6.25$ max

$A(10) = \frac{100}{4\pi} \approx \boxed{7.95}$

$A\left(\frac{10\pi}{4+\pi}\right) = \boxed{3.5}$ min

conclusion:
10 m of wire used for the circle to maximize the enclosed area.

$\frac{10\pi}{4+\pi}$ m will be bent into a circle & $10 - \frac{10\pi}{4+\pi}$ into a square to minimize the enclosed area.