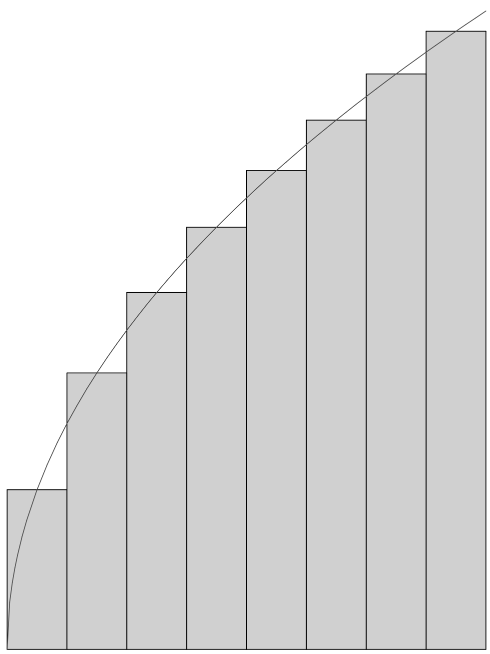
Section 6.2: Area

Using rectangles to approximate the area under a curve Let $f(x)$ be a function defined on the interval $[a, b]$. We wish to approximate the area bounded by the curve $f(x)$, the x-axis, $x = a$ and $x = b$. We begin by partitioning the interval $[a, b]$ into n smaller subintervals. We call $P = \{a = x_0, x_1, x_2, ..., x_{n-1}, x_n = b\}$, where $a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b$, the partition points. For each subinterval $[x_{i-1}, x_i]$, choose a representative point x_i^*, that is x_i^* is any point on the interval $[x_{i-1}, x_i]$. For each subinterval $[x_{i-1}, x_i]$, we will construct a rectangle under the curve and above the x-axis, where the height of this rectangle is $f(x_i^*)$ and the width is $\Delta x_i = x_i - x_{i-1}$. Refer to the figure below.



As seen from the above figure, the sum of the approximating rectangles gives an approximation under the graph of $f(x)$ from $x = a$ to $x = b$. Moreover, the area under the curve $\approx \sum_{i=1}^{n} f(x_i^*)\Delta x_i$, where n is the number of rectangles constructed.

EXAMPLE 1: You are given a function f, an interval, partition points, and a description of the point x_i^* within the ith subinterval.
(a) Find $||P||$.

(b) Sketch the graph of f and the approximating rectangles.

(c) Find the sum of the approximating rectangles.

(i) $f(x) = 16 - x^2$, $[0, 4]$, $P = \{0, 1, 2, 3, 4\}$, $x_i^* =$ left endpoint.

(ii) $f(x) = 4\cos x$, $[0, \frac{\pi}{2}]$, $P = \{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\}$, $x_i^* =$ Right endpoint.

(iii) $f(x) = \ln x$, $[1, 3]$. Create the partition by using 3 subintervals of equal length, $x_i^* =$ midpoint.

Theorem If $f(x) \geq 0$ on the interval $[a, b]$, then the true area under the graph of $f(x)$ from $x = a$ to $x = b$ is $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$, where $\Delta x_i = \dfrac{b-a}{n}$ and x_i^* is any point on the ith subinterval.

EXAMPLE 2: For the following functions, set up the limit of the Reimann Sum that represents the area under the graph of $f(x)$ on the given interval. Do not evaluate the limit.

(i) $f(x) = x^2 + 3x - 2$ on the interval $[1, 4]$.

(ii) $f(x) = \sqrt{x^2 + 1}$ on the interval $[0, 5]$.

EXAMPLE 3: The following limits represent the area under the graph of $f(x)$ from $x = a$ to $x = b$. Identify $f(x)$, a and b.

(i) $\displaystyle\lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \sqrt{1 + \frac{3i}{n}}$

(ii) $\displaystyle\lim_{n\to\infty} \frac{10}{n} \sum_{i=1}^{n} \frac{1}{1 + \left(7 + \frac{10}{n} i\right)^3}$