Using rectangles to approximate the area under a curve. Let $f(x)$ be a function defined on the interval $[a, b]$. We wish to approximate the area bounded by the curve $f(x)$, the $x$-axis, $x = a$ and $x = b$. We begin by partitioning the interval $[a, b]$ into $n$ smaller subintervals. We call $P = \{a = x_0, x_1, x_2, ..., x_{n-1}, x_n = b\}$, where $a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b$, the partition points. For each subinterval $[x_{i-1}, x_i]$, choose a representative point $x_i^*$, that is $x_i^*$ is any point on the interval $[x_{i-1}, x_i]$. For each subinterval $[x_{i-1}, x_i]$, we will construct a rectangle under the curve and above the $x$-axis, where the height of this rectangle is $f(x_i^*)$ and the width is $\Delta x_i = x_i - x_{i-1}$. Refer to the figure below.

As seen from the above figure, the sum of the approximating rectangles gives an approximation under the graph of $f(x)$ from $x = a$ to $x = b$. Moreover, the area under the curve $\approx \sum_{i=1}^{n} f(x_i^*) \Delta x_i$, where $n$ is the number of rectangles constructed.
EXAMPLE 1: You are given a function \( f \), an interval, partition points, and a description of the point \( x_i^* \) within the \( i \)th subinterval.

(a) Find \( ||P|| \).

(b) Sketch the graph of \( f \) and the approximating rectangles.

(c) Find the sum of the approximating rectangles.

(i) \( f(x) = 16 - x^2 \), \([0, 4]\), \( P = \{0, 1, 2, 3, 4\} \), \( x_i^* = \text{left endpoint} \).

\[
\begin{align*}
(a) \quad ||P|| &= \max \{ \Delta x_i \} = 1 \\
(c) \quad \sum_{i=1}^{4} f(x_i^*)(\Delta x_i) &= f(0)(1) + f(1)(1) + f(2)(1) + f(3)(1) \\
&= 16 + 15 + 12 + 7 \\
&= 50 \\
\text{L}_4 &= 50 \quad \text{over estimate}
\end{align*}
\]
(ii) \( f(x) = 4 \cos x \), \([0, \frac{\pi}{2}]\), \( P = \{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\} \), \( x^*_i = \text{Right endpoint.} \)

\[
\text{(b) } \| \rho \| = \max \{ \delta x_i \} = \frac{\pi}{6}
\]

\[
\text{(c) } \sum_{i=1}^{4} f(x^*_i) \delta x_i
\]

\[
f(x) = 4 \cos x = f\left(\frac{\pi}{6}\right) + f\left(\frac{\pi}{4}\right) + f\left(\frac{\pi}{3}\right) + f\left(\frac{\pi}{2}\right)
\]

\[
k_4 = 4 \cdot \frac{\sqrt{2}}{2} + 4 \cdot \frac{\sqrt{3}}{2} + 4 \cdot \frac{1}{2} + 4 \cdot 0 = \frac{10\pi}{6}
\]

Under approximation
(iii) \( f(x) = \ln x \), \([1, 3]\). Create the partition by using 3 subintervals of equal length, \( x_i^* = \text{midpoint}. \)

If subintervals have equal length

\[
\Delta x = \frac{b-a}{n}
\]

\[
a = \text{far left end point}
\]
\[
b = \text{right end point}
\]
\[
n = \text{number of subintervals}
\]

Here, \( a = 1, \ b = 3, \ n = 3 \Rightarrow \Delta x = \frac{3-1}{3} = \frac{2}{3} \)

\[
\{ \frac{1}{3}, \frac{2}{3}, \frac{3}{3} \}
\]

\[
\text{(a) } \| P' \| = \frac{2}{3}
\]

\[
\text{(b) }
\]

\[
\text{(c) } \sum_{i=1}^{3} f(x_i^*) \Delta x
\]

\[
= f\left(\frac{4}{3}\right)\left(\frac{2}{3}\right) + f(2)\left(\frac{2}{3}\right) + f\left(\frac{8}{3}\right)\left(\frac{2}{3}\right)
\]

\[
= \left(\ln \frac{4}{3} + \ln 2 + \ln \frac{8}{3}\right)\left(\frac{2}{3}\right)
\]
**Example 2:** For the following functions, set up the limit of a Riemann Sum that represents the area under the graph of \( f(x) \) on the given interval. Do not evaluate the limit!

(i) \( f(x) = x^2 + 3x - 2 \) on the interval \([1, 4]\) using right endpoints.

**Theorem:**

\[
A_{\text{curve}} = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x
\]

1. Divide \([1, 4]\) into \( n \) equal sub-intervals

\[
\Delta x = \frac{b-a}{n} = \frac{4-1}{n} = \frac{3}{n}
\]

2. Find a formula for \( x_i^* = \text{right endpoint} \)

<table>
<thead>
<tr>
<th>( x_0 )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1 + ( \frac{3}{n} )</td>
<td>1 + ( \frac{3}{n} )(2)</td>
<td>1 + ( \frac{3}{n} )(3)</td>
<td>1 + ( \frac{3}{n} )(4)</td>
<td>( 1 + \frac{3}{n} )(i)</td>
</tr>
</tbody>
</table>

**General formula:** \( x_i = a + (b-x) i \)

\[
A = \lim_{n \to \infty} \sum_{i=1}^{n} f(1 + \frac{3}{n} i) \left( \frac{3}{n} \right)
\]

\[
= \lim_{n \to \infty} \sum_{i=1}^{n} \left[ (1 + \frac{3}{n} i)^2 + 3(1 + \frac{3}{n} i) - 2 \right] \left( \frac{3}{n} \right)
\]
(ii) \( f(x) = \sqrt{x^2 + 1} \) on the interval \([0, 5]\) using right endpoints.

\[
\begin{align*}
A &= \lim_{n \to \infty} \sum_{i=1}^{n} f \left( a + (\Delta x) i \right) \Delta x \\
&= \lim_{n \to \infty} \sum_{i=1}^{n} f \left( 0 + \frac{5}{n} i \right) \left( \frac{5}{n} \right) \\
&= \lim_{n \to \infty} \sum_{i=1}^{n} \left( \sqrt{\left( \frac{5}{n} i \right)^2 + 1} \right) \left( \frac{5}{n} \right)
\end{align*}
\]
EXAMPLE 3: The following limits represent the area under the graph of \( f(x) \) from \( x = a \) to \( x = b \). Identify \( f(x) \), \( a \) and \( b \).

(i) \( \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{1 + \frac{3i}{n}} = \lim_{n \to \infty} \sum_{i=1}^{n} f \left( a + \left( \frac{b-a}{n} \right) i \right) \left( \frac{b-a}{n} \right) \)

Find \( f(x) \), \( a \) and \( b \)

\[ 1 + \frac{3i}{n} = a + \left( \frac{b-a}{n} \right) i \]

\[ a = 1 \]
\[ b = 4 \]
\[ f(x) = \sqrt{x} \]

(ii) \( \lim_{n \to \infty} \frac{10}{n} \sum_{i=1}^{n} \frac{1}{1 + \left( \frac{7 + \frac{10i}{n}}{n} \right)} \)

\[ \Delta x = \frac{10}{n} \]

\[ f(x) = \frac{1}{1 + x^3} \]

\( \Delta x \) as defined above:

\[ f(x) = \sqrt{1 + x} \]

\( a = 0 \), \( b = 3 \)