Section 6.3: The Definite Integral

Recall If $f(x) \geq 0$ on the interval $[a, b]$, the area under the curve of $f(x)$, above the x-axis, from $x = a$ to $x = b$ is $\approx \sum_{i=1}^{n} f(x_i^*)\Delta x_i$. We call this sum a **Riemann Sum.**

EXAMPLE 1: If $f(x) = x + x^2$, the interval $[0, 4]$, partition points $P = \{0, 1, 2, 3, 4\}$, and x_i^* = left endpoint. Find the Riemann Sum.

Recall If $f(x) \geq 0$ on the interval $[a, b]$, then the true area under the graph of $f(x)$ from $x = a$ to $x = b$ is $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$, where $\Delta x_i = \dfrac{b-a}{n}$ and x_i^* is any point on the ith subinterval. We would like to define the limit of a Riemann Sum irregardless of whether the function is positive. To that end, we will define the definite integral.

Definition The Definite Integral of $f(x)$ from $x = a$ to $x = b$ is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

where $\Delta x_i = \dfrac{b-a}{n}$ and x_i^* is any point on the ith subinterval. In the event $f(x)$ is positive on the interval $[a, b]$, then the definite integral is the same as the area bounded by $f(x)$, the x-axis, $x = a$ and $x = b$. If $f(x)$ is not always positive on the interval $[a, b]$, then the definite integral is the net area.

Fact Any Riemann sum can approximate a definite integral. Specifically, the **Midpoint Rule** can be used to approximate the definite integral.

Midpoint Rule: $\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\overline{x_i})\Delta x$, where $\Delta x = \frac{b-a}{n}$ and $\overline{x_i}$ is the midpoint of the ith subinterval.

EXAMPLE 2: Use the Midpoint Rule with $n = 4$ to approximate $\int_1^5 \sqrt{x^2+1}\,dx$.

EXAMPLE 3: Use Geometry to evaluate the following definite integrals:

(i) $\int_0^3 (1-2x)\,dx$

(ii) $\int_{-1}^3 |x-2|\,dx$

(iii) $\int_{-2}^0 \sqrt{4-x^2}\,dx$

Theorem

(a) $\int_a^b c\,dx = c(b-a)$

(b) $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$

(c) $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

(d) $\int_a^a f(x)\,dx = 0$

(e) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(f) $\int_a^b dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

(g) If $m \leq f(x) \leq M$ for all x in the interval $[a, b]$, then
$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$.

EXAMPLE 4: Find $\int_1^1 \sqrt{x^5 + x^2 + 1}\,dx$

EXAMPLE 5: If $\int_1^3 f(x)\,dx = 4$ and $\int_1^3 g(x)\,dx = -3$, find $\int_1^3 (f(x) + 2g(x))\,dx$.

EXAMPLE 6: Write $\int_{-3}^5 f(x)\,dx - \int_{-3}^0 f(x)\,dx + \int_5^6 f(x)\,dx$ as a single integral.

EXAMPLE 7: Find an upper and lower bound on $\int_0^2 \sqrt{x^3+1}\,dx$.

EXAMPLE 8: Express the following limits as a definite integral:

(i) $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1+(i/n)^2}$

(ii) $\lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left(3(1+\frac{2i}{n}))^5 - 6\right)$