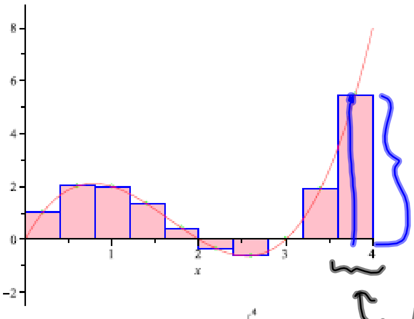
Section 6.4: The Fundamental Theorem of Calculus

Recall $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$ where $\Delta x_i = \dfrac{b-a}{n}$ and x_i^* is any point on the ith subinterval. In the event $f(x)$ is positive on the interval $[a,b]$, then the definite integral is the same as the area bounded by $f(x)$, the x-axis, $x = a$ and $x = b$. If $f(x)$ is not always positive on the interval $[a,b]$, then the definite integral is the net area.

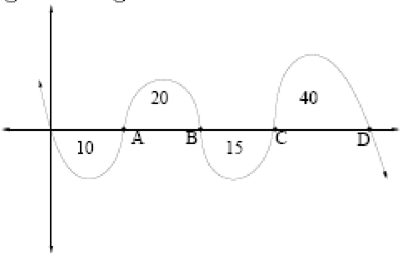
For the graph provided below, $\int_0^4 f(x)\,dx \approx \sum_{i=1}^{10} f(x_i^*)\Delta x_i$, where x_i is the midpoint of the ith subinterval.

$$\sum_{i=1}^{n} \underbrace{f(x_i^*)}_{\text{height}} \underbrace{\Delta x_i}_{\text{width}}$$



A midpoint Riemann sum approximation of $\int_0^4 f(x)\,dx$, where $f(x) = x\,(x-2)\,(x-3)$ and the partition is uniform. The approximate value of the integral is 5.280000000. Number of partitions used: 10.

EXAMPLE 1: Refer to the graph provided below to compute the following definite integrals using the indicated areas:



$\int_0^A f(x)\,dx = -10$

$\int_0^C f(x)\,dx = -10 + 20 - 15 = 20 - 25$
$= -5$

$\int_A^B f(x)\,dx = 20$

$\int_0^D f(x)\,dx = 35$

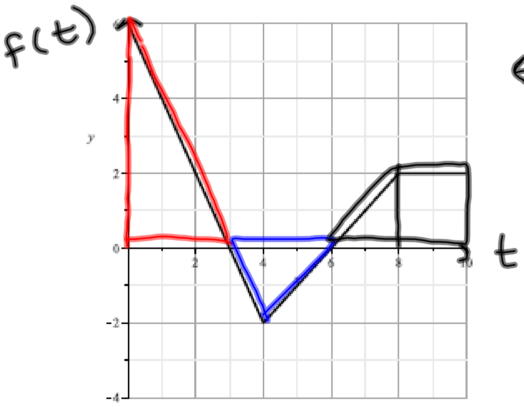
$\int_C^A f(x)\,dx = -\int_A^C f(x)\,dx$
$= -[20 - 15]$
$= \boxed{-5}$

$\int_C^C f(x)\,dx = 0$

The Fundamental Theorem of Calculus

- Part I: If f is continuous on $[a, b]$, then the function g defined by $g(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b) and $g'(x) = f(x)$.
- Part II: If f is continuous on $[a, b]$, then $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is an antiderivative of f.

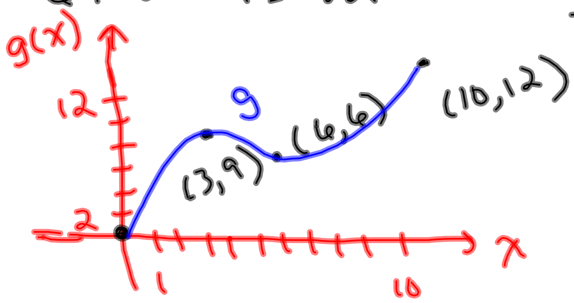
EXAMPLE 2: If $g(x) = \int_0^x f(t)\,dt$, where the graph of $f(t)$ is given below, where $0 \le x \le 10$, evaluate $g(0)$, $g(3)$, $g(6)$ and $g(10)$. Where is $g(x)$ increasing? What is the maximum value of $g(x)$?



← f(t)

$g(x) = \int_0^x f(t)\,dt$

- $g(0) = \int_0^0 f(t)\,dt = 0$
- $g(3) = \int_0^3 f(t)\,dt = \frac{1}{2}(3)(6) = 9$
- $g(6) = g(3) + \int_3^6 f(t)\,dt = 9 - \frac{1}{2}(3)(2) = 6$
- $g(10) = g(6) + \int_6^{10} f(t)\,dt = 6 + \frac{1}{2}(2)(2) + 2(2) = 6 + 2 + 4 = 12$



- g increasing $0 < x < 3$, $6 < x < 10$
- max value of $g = 12$

EXAMPLE 3: Find the derivative of:

(a) $g(x) = \int_{-1}^{x} \sqrt{t^3+1}\,dt$

Let $F(t)$ be an antiderivative of $f(t)$ then $\underbrace{F'(t) = f(t)}$

$g(x) = F(t)\Big|_{-1}^{x}$

$g(x) = F(x) - F(-1)$

$g'(x) = F'(x) - 0$

$= f(x)$

$\boxed{g'(x) = \sqrt{x^3+1}}$

FACT:

$g(x) = \int_a^x f(t)\,dt$

$g'(x) = f(x)$

(b) $g(x) = \int_1^{\sqrt{x}} \underbrace{\frac{t^2}{t^2+1}}_{f(t)}\,dt \Rightarrow g(x) = F(t)\Big|_1^{\sqrt{x}}$

chain rule

$g(x) = F(\sqrt{x}) - F(1)$

$g'(x) = F'(\sqrt{x})\left(\frac{1}{2}x^{-\frac{1}{2}}\right) - 0$

$= f(\sqrt{x})\left(\frac{1}{2\sqrt{x}}\right)$

$= \boxed{\frac{x}{x+1} \cdot \frac{1}{2\sqrt{x}}}$

(c) $g(x) = \int_{x^2}^{\sin x} \frac{\cos t}{t}\,dt$

$g'(x) = \frac{\cos(\sin x)}{\sin x} \cdot \cos x - \frac{\cos(x^2)}{x^2}(2x)$

EXAMPLE 4:

(a) Evaluate $\int_1^2 \frac{1}{x^2}\,dx$ $= \int_1^2 x^{-2}\,dx$

$= \frac{x^{-1}}{-1}\Big|_1^2 \rightarrow -\frac{1}{x}\Big|_1^2$

$= -\frac{1}{2} - (-1)$

$= \boxed{\frac{1}{2}}$

(b) Evaluate $\int_{-1}^{0} (5x^2 - 4x + 3)\,dx$

$\left(\frac{5x^3}{3} - \frac{4x^2}{2} + 3x\right)\Big|_{-1}^{0}$

$= 0 - \left(-\frac{5}{3} - 2 - 3\right) = \frac{5}{3} + 5 = \boxed{\frac{20}{3}}$

(c) Evaluate $\int_{\ln 3}^{\ln 6} 8e^t\,dt$

$\int_{\ln 3}^{\ln 6} 8e^t\,dt = 8e^t\Big|_{\ln 3}^{\ln 6}$

$= 8\cancel{e^{\ln 6}}^{\,6} - 8\cancel{e^{\ln 3}}^{\,3}$

$= 48 - 24$

$= \boxed{24}$

Added problem

c') $\int_0^1 3^x\,dx$

$= \frac{3^x}{\ln 3}\Big|_0^1$

$= \boxed{\frac{3}{\ln 3} - \frac{1}{\ln 3}}$

(d) Evaluate $\int_1^2 \frac{x^6 - x^2}{x^7}\,dx$ $= \int_1^2 \left(\frac{x^6}{x^7} - \frac{x^2}{x^7} \right) dx$

subtract powers

$$= \int_1^2 \left(x^{-1} - x^{-5} \right) dx$$

$$= \left(\ln|x| - \frac{x^{-4}}{-4} \right) \Big|_1^2$$

$$= \left(\ln x + \frac{1}{4x^4} \right) \Big|_1^2 = \ln 2 + \frac{1}{64} - \left(\cancel{\ln 1}^{\,0} + \frac{1}{4} \right)$$

$$= \boxed{\ln 2 + \frac{1}{64} - \frac{1}{4}}$$

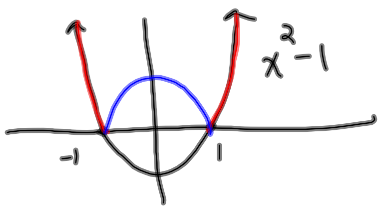
(e) Evaluate $\int_0^2 (w^3 - 1)^2\,dw$

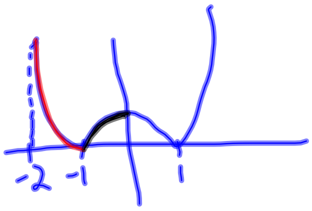
$$\int_0^2 \left(w^6 - 2w^3 + 1 \right) dw$$

$$\left(\frac{w^7}{7} - \frac{\cancel{2}w^4}{\cancel{4}\,2} + w \right) \Big|_0^2 = \frac{128}{7} - 8 + 2 - 0$$

$$= \boxed{\frac{128}{7} - 6}$$

(f) Evaluate $\int_{-2}^{0} |x^2 - 1|\,dx$





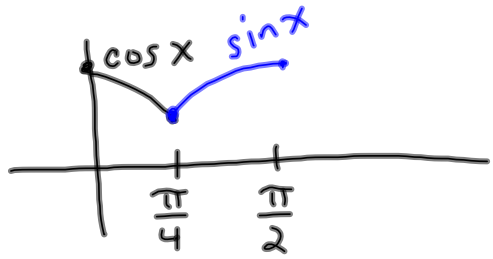
$$\int_{-2}^{0} |x^2 - 1|\,dx =$$

$$\int_{-2}^{-1} (x^2 - 1)\,dx + \int_{-1}^{0} (1 - x^2)\,dx$$

$$\left(\frac{x^3}{3} - x \right) \Big|_{-2}^{-1} + \left(x - \frac{x^3}{3} \right) \Big|_{-1}^{0}$$

$$= \boxed{2}$$

(g) Find $\int_0^{\pi/2} f(x)\,dx$ where $f(x) = \begin{cases} \cos x & \text{if } 0 \le x < \frac{\pi}{4} \\ \sin x & \text{if } \frac{\pi}{4} \le x \le \frac{\pi}{2} \end{cases}$

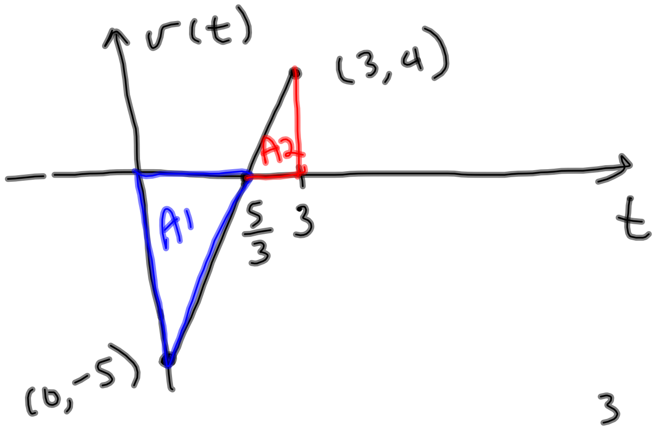


$$\int_0^{\frac{\pi}{4}} \cos x\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x\,dx$$

$$\sin x \Big|_0^{\frac{\pi}{4}} + -\cos x \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{\sqrt{2}}{2} - 0 + -0 + \frac{\sqrt{2}}{2}$$

$$= \boxed{\sqrt{2}}$$

EXAMPLE 5: Suppose an object is moving according to velocity $v(t) = 3t - 5$, $0 \le t \le 3$. Find the displacement and distance traveled during the first 3 seconds.



$$v(t) = 0$$
$$3t - 5 = 0$$
$$3t = 5$$
$$t = \frac{5}{3}$$

$$\text{displacement} = \int_0^3 v(t)\,dt$$

$$\text{distance} = \int_0^3 |v(t)|\,dt$$

$$A_1 = \frac{1}{2}\left(\frac{5}{3}\right)(5)$$
$$A_1 = \frac{25}{6}$$
$$A_2 = \frac{1}{2}\left(\frac{4}{3}\right)(4)$$
$$A_2 = \frac{16}{6}$$

$$\text{displacement} = -A_1 + A_2$$
$$= -\frac{25}{6} + \frac{16}{6} = -\frac{9}{6} = -\frac{3}{2} \text{ units}$$

$$\text{distance} = A_1 + A_2 = \frac{25}{6} + \frac{16}{6} = \frac{41}{6} \text{ units}$$