Section 6.4: The Fundamental Theorem of Calculus

Recall \( \int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x_i \) where \( \Delta x_i = \frac{b - a}{n} \) and \( x_i^* \) is any point on the \( i \)th subinterval. In the event \( f(x) \) is positive on the interval \([a, b]\), then the definite integral is the same as the area bounded by \( f(x) \), the \( x \)-axis, \( x = a \) and \( x = b \). If \( f(x) \) is not always positive on the interval \([a, b]\), then the definite integral is the net area.

For the graph provided below, \( \int_a^b f(x) \, dx \approx \sum_{i=1}^{10} f(x_i^*) \Delta x_i \), where \( x_i \) is the midpoint of the \( i \)th subinterval.

\[ \int_a^b f(x) \, dx = \text{net Area} \]

\[ A = f(x) \Delta x \]

\[ A = \int_a^b f(x) \, dx \]

\[ \int_a^b f(x) \, dx = \text{net Area} = A_1 - A_2 \]

**EXAMPLE 1:** Refer to the graph provided below to compute the following definite integrals using the indicated areas:

\[ \int_a^A f(x) \, dx = -10 \]
\[ \int_a^B f(x) \, dx = 20 \]
\[ \int_A^C f(x) \, dx = -\int_A^C f(x) \, dx \]
\[ \int_C^D f(x) \, dx = 0 \]
\[ \int_a^C f(x) \, dx = \int_C^D f(x) \, dx = -5 + 40 = 35 \]
\[ \int_a^D f(x) \, dx = -10 + 20 - 15 = -5 \]

\[ \int_C^D f(x) \, dx = 0 \]

\[ \int_A^C f(x) \, dx = - (20 - 15) = -5 \]
The Fundamental Theorem of Calculus

- Part I: If $f$ is continuous on $[a, b]$, then the function $g$ defined by
  $g(x) = \int_a^x f(t) \, dt$ is continuous on $[a, b]$ and differentiable on $(a, b)$ and
  $g'(x) = f(x)$.

- Part II: If $f$ is continuous on $[a, b]$, then
  $\int_a^b f(x) \, dx = F(b) - F(a)$, where $F$ is an antiderivative of $f$.

**EXAMPLE 2:** If $g(x) = \int_0^x f(t) \, dt$, where the graph of $f(t)$ is given below, where
$0 \leq x \leq 10$, evaluate $g(0), g(3), g(6)$ and $g(10)$. Where is $g(x)$ increasing? What is the
maximum value of $g(x)$?

\[
\begin{align*}
g(0) &= \int_0^0 f(t) \, dt = 0 \\
g(3) &= \int_0^3 f(t) \, dt = \frac{1}{3}(3)(6) = 9 \\
g(6) &= 9 - \frac{1}{3}(3)(2) = 6 \\
g(10) &= 6 + \frac{1}{2}(2)(2) + 2(2) = 12
\end{align*}
\]

$g$ increasing for $0 < x < 3$

$6 < x < 10$

absolute maximum of $g(x)$ is 12
EXAMPLE 3: Find the derivative of:

(a) \( g(x) = \int_{-1}^{x} \sqrt{t^3 + 1} \, dt \)

\[
\frac{d}{dx} \int_{-1}^{x} \sqrt{t^3 + 1} \, dt = \sqrt{x^3 + 1}
\]

(b) \( g(x) = \int_{1}^{\sqrt{x}} \frac{t^2}{t^2 + 1} \, dt \)

\[
g'(x) = \left( \frac{\sqrt{x}}{(\sqrt{x})^2 + 1} \right)^2 \cdot \frac{1}{2} \sqrt{x} = \frac{\sqrt{x}}{x + 1} \cdot \frac{1}{2} \sqrt{x}
\]

(c) \( g(x) = \int_{12}^{\sin x} \frac{\cos t}{t} \, dt \)

\[
g'(x) = \frac{\cos(\sin x)}{\sin x} \cdot \cos x - \cos(x^2) \cdot 2x
\]
EXAMPLE 4:

(a) Evaluate \( \int_{1}^{2} \frac{1}{x^2} \, dx \)

\[
\int_{2}^{1} x^{-2} \, dx = \left. \frac{x^{-1}}{-1} \right|_{2}^{1} = \left. \frac{-1}{x} \right|_{1}^{2} = \frac{-1}{2} - (-1) = \frac{1}{2}
\]

(b) Evaluate \( \int_{-1}^{0} (5x^2 - 4x + 3) \, dx = \left( \frac{5x^3}{3} - \frac{4x^2}{2} + 3x \right) \right|_{-1}^{0} = 0 - \left( \frac{-5}{3} - 2 - 3 \right) = \frac{20}{3}

(c) Evaluate \( \int_{ln3}^{ln6} 8e^t \, dt \)

Recall: \( e^{\ln x} = x \)

\[
\int_{ln3}^{ln6} 8e^t \, dt = 8e^{ln6} - 8e^{ln3} = 8(6) - 8(3) = 24
\]

\[
\int_{2}^{3} 5 \cdot (4^x) \, dx = \left. \frac{5}{\ln 4} \cdot 4^x \right|_{2}^{3} = \frac{5}{\ln 4} (4^3 - 4^2)
\]
(d) Evaluate \( \int_1^2 \frac{x^6 - x^2}{x^7} \, dx \)
\[
= \left. \int_1^2 \left( \frac{x^6}{x^7} - \frac{x^2}{x^7} \right) \, dx \right|_1^2 \\
= \left. \left( \ln|x| - \frac{x^5}{5} \right) \right|_1^2 \\
= \ln 2 + \frac{1}{6} - (\ln 1 + \frac{1}{4}) \\
= \ln 2 - \frac{15}{64}
\]

(e) Evaluate \( \int_0^2 (w^3 - 1)^2 \, dw \)
\[
\int_0^2 (w^6 - 2w^3 + 1) \, dw = \left( \frac{w^7}{7} - \frac{2w^4}{4} + w \right) \bigg|_0^2 \\
= \frac{128}{7} - 8 + 2
\]

(f) Evaluate \( \int_{-2}^0 |x^2 - 1| \, dx \)
\[
= \left. \int_{-2}^0 (x^2 - 1) \, dx \right|_{-2}^0 + \left. \int_{-2}^1 (-x^2 + 1) \, dx \right|_{-2}^0 \\
= \left. \left( \frac{x^3}{3} - x \right) \right|_{-2}^0 + \left. \left(-\frac{x^3}{3} + x \right) \right|_{-2}^0 \\
= 2
\]
(g) Find \( \int_{0}^{\pi/2} f(x) \, dx \) where 
\[
 f(x) = \begin{cases} 
 \cos x & \text{if } 0 \leq x < \frac{\pi}{4} \\
 \sin x & \text{if } \frac{\pi}{4} \leq x \leq \frac{\pi}{2}
\end{cases}
\]

\[
\int_{0}^{\pi/2} f(x) \, dx = \int_{0}^{\pi/4} \cos x \, dx + \int_{\pi/4}^{\pi/2} \sin x \, dx
\]

\[
= \sin \left( \frac{\pi}{4} \right) - \cos \left( \frac{\pi}{4} \right)
\]

\[
= \frac{\sqrt{2}}{2} - 0 - (0 - \frac{\sqrt{2}}{2})
\]

\[
= \sqrt{2}
\]

**EXAMPLE 5:** Suppose an object is moving according to velocity \( v(t) = 3t - 5 \), 
\( 0 \leq t \leq 3 \). Find the displacement and distance traveled during the first 3 seconds.

\[
v(t) = 0 \rightarrow 3t - 5 = 0 \quad \Rightarrow \quad t = \frac{5}{3}
\]

\[
A_1 = \frac{1}{3} (\frac{5}{3}) (5) = \frac{25}{6}
\]

\[
A_2 = \frac{1}{2} (\frac{4}{3}) (4) = \frac{16}{6}
\]

**Displacement:** 
\[
\int_{0}^{3} v(t) \, dt = -A_1 + A_2 \quad \Rightarrow \quad -\frac{25}{6} + \frac{16}{6} = -\frac{9}{6}
\]

**Distance:** 
\[
\int_{0}^{3} |v(t)| \, dt = A_1 + A_2 \quad \Rightarrow \quad \frac{25}{6} + \frac{16}{6} = \frac{41}{6}
\]