Section 6.5: Integration by Substitution

The Substitution Rule: If $u = g(x)$ is a differentiable function, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

EXAMPLE 1: Integrals of the form $\int (f(x))^n f'(x)\,dx$ $\quad u = f(x) \quad du = f'(x)\,dx$

a.) $\int x\underbrace{(x^2+3)}_{u}{}^{10}\,dx \quad \frac{du}{2x}$

$\boxed{u = x^2+3}$

$du = 2x\,dx$

$\boxed{\frac{du}{2x} = dx}$

$$\int \cancel{x}\,u^{10}\frac{du}{2\cancel{x}} = \frac{1}{2}\int u^{10}\,du$$

$$= \frac{1}{2}\,\frac{u^{11}}{11} + C$$

$$\boxed{= \frac{1}{22}(x^2+3)^{11} + C}$$

b.) $\int \frac{2t^2}{\sqrt{\underbrace{t^3-1}_{u}}}\,dt \quad \frac{du}{3t^2}$

$\boxed{u = t^3-1}$

$du = 3t^2\,dt \Rightarrow \boxed{\frac{du}{3t^2} = dt}$

$$\int \frac{2\cancel{t^2}}{\sqrt{u}}\,\frac{du}{3\cancel{t^2}} = \frac{2}{3}\int u^{-\frac{1}{2}}\,du$$

$$= \frac{2}{3}\,\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= \frac{4}{3}(t^3-1)^{\frac{1}{2}} + C$$

$$\boxed{= \frac{4}{3}\sqrt{t^3-1} + C}$$

c.) $\int_0^1 \frac{x}{(x+1)^2}\,dx$

$=$

$\boxed{u = x+1}$ — $x=1,\ u=2$; $x=0,\ u=1$

$du = dx$

$x = u-1$

$$\int_1^2 \frac{x}{u^2}\,du$$

$$\int_1^2 \frac{u-1}{u^2}\,du \rightarrow \int_1^2 \left(\frac{1}{u} - \frac{1}{u^2}\right) du$$

$$\int_1^2 \left(\frac{1}{u} - u^{-2}\right) du$$

$$= \left(\ln|u| - \frac{u^{-1}}{-1}\right)\Big|_1^2$$

$$= \left(\ln u + \frac{1}{u}\right)\Big|_1^2$$

$$= \ln 2 + \frac{1}{2} - (\cancelto{0}{\ln 1} + 1)$$

$$= \boxed{\ln 2 - \frac{1}{2}}$$

EXAMPLE 2: Integrals of the form $\int e^{f(x)} f'(x)\,dx$

a.) $\int_{-1}^{2} x e^{x^2}\,dx$

$u = x^2$ — $x=2,\ u=4$; $x=-1,\ u=1$

$du = 2x\,dx$

recall: $\int e^x\,dx = e^x + C$

just x

$$= \int_1^4 \cancel{x} e^u \frac{du}{2\cancel{x}} = \frac{1}{2}\int_1^4 e^u\,du$$

$$= \frac{1}{2} e^u \Big|_1^4$$

$$= \boxed{\frac{1}{2}\left[e^4 - e\right]}$$

EXAMPLE 3: Integrals of the form $\int \frac{f'(x)}{f(x)}\,dx$

a.) $\int_0^1 \frac{x}{x^2+1}\,dx$

$u = x^2+1$; $x=1,\ u=2$; $x=0,\ u=1$

$du = 2x\,dx$

$$\int_1^2 \frac{x}{u}\frac{du}{2x} = \frac{1}{2}\int \frac{du}{u}\Big|_1^2$$

$$= \frac{1}{2}\ln u\Big|_1^2$$

$$= \frac{1}{2}\left(\ln(2) - \cancel{\ln(1)}^{\,0}\right)$$

$$= \boxed{\frac{1}{2}\ln 2 = \ln 2^{\frac{1}{2}} = \ln\sqrt{2}}$$

b.) $\int \frac{ax+b}{ax^2+2bx+c}\,dx$

$u = ax^2 + 2bx + c$

$du = (2ax + 2b)\,dx$

$du = 2(ax+b)\,dx$

$$\int \frac{\cancel{ax+b}}{u}\frac{du}{2\cancel{(ax+b)}} = \frac{1}{2}\int\frac{du}{u}$$

$$= \frac{1}{2}\ln|u| + C$$

$$\boxed{= \frac{1}{2}\ln\left|ax^2+2bx+c\right| + C}$$

c.) $\int \tan(x)\,dx = \int \frac{\sin x}{\cos x}\,dx$

$u = \cos x$

$du = -\sin x\,dx$

$\frac{du}{-\sin x} = dx$

$$\int \frac{\sin x}{u}\frac{du}{-\sin x}$$

$$= -\int \frac{du}{u}$$

$$= -\ln|u| + C$$

$\rightarrow -\ln|\cos x| + C$

$\ln|\sec x| + C$

$$\boxed{\int \tan x\,dx = \ln|\sec x| + C}$$

d.) $\int_{e^2}^{e^3} \frac{1}{x \ln x}\,dx$

~~$u = x\ln x$~~

~~$du = (\ln x + 1)\,dx$~~

$$\int_{e^2}^{e^3} \frac{\frac{1}{x}}{\ln x}\,dx$$

du

u

$u = \ln x$ — $x = e^3,\ u = \ln e^3 = 3$; $x = e^2,\ u = \ln e^2 = 2$

$du = \frac{1}{x}\,dx$

$$\int_2^3 \frac{du}{u} = \ln u \Big|_2^3 = \ln 3 - \ln 2$$

$$= \boxed{\ln \frac{3}{2}}$$

EXAMPLE 4: Integrals of the form $\int \cos(f(x))f'(x)\,dx$, $\int \sin(f(x))f'(x)\,dx$, $\int \sec^2(f(x))f'(x)\,dx$, $\int \sec(f(x))\tan(f(x))f'(x)\,dx$, $\int \csc^2(f(x))f'(x)\,dx$, $\int \csc(f(x))\cot(f(x))f'(x)\,dx$

a.) $\int x\cos(2-x^2)\,dx$

$u =$ angle

$u = 2 - x^2$

$du = -2x\,dx$

$\frac{du}{-2x} = dx$

$$\int \cancel{x}\cos(u)\left(\frac{du}{-2\cancel{x}}\right)$$

$$-\frac{1}{2}\int \cos u\,du = -\frac{1}{2}\sin u + C$$

$$= \boxed{-\frac{1}{2}\sin(2-x^2) + C}$$

b.) $\int \frac{\sec^2\sqrt{x}}{\sqrt{x}}\,dx$

$u = \sqrt{x} = x^{\frac{1}{2}}$

$du = \frac{1}{2}x^{-\frac{1}{2}}\,dx$

$du = \frac{1}{2\sqrt{x}}\,dx \Rightarrow 2\sqrt{x}\,du = dx$

$$\int \frac{\sec^2 u}{\cancel{\sqrt{x}}}\,2\cancel{\sqrt{x}}\,du = 2\int \sec^2 u\,du$$

$$= 2\tan u + C$$

$$= \boxed{2\tan\sqrt{x} + C}$$