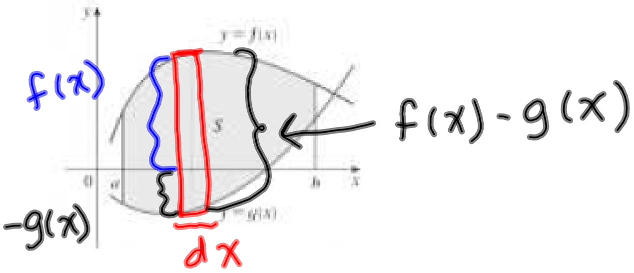
Section 7.1: Area

Area

- The area bounded by the curves $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $x = b$, where $f(x) \geq g(x)$ for all x in the interval $[a, b]$ is

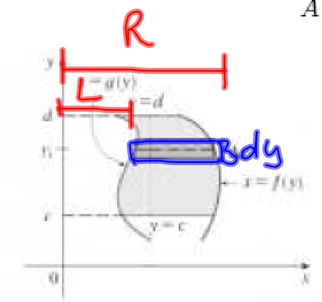
$$S = \int_a^b (f(x) - g(x))\, dx = \int_a^b (\text{Top} - \text{Bottom})\, dx$$



- The area bounded by the curves $x = f(y)$, $x = g(y)$ and the lines $y = c$ and $y = d$, where $f(y) \geq g(y)$ for all y in the interval $[c, d]$ is

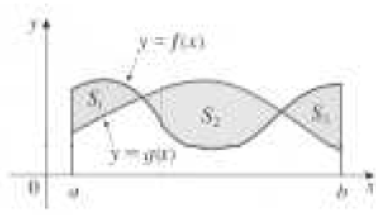
$$A = \int_c^d (f(y) - g(y))\, dy$$

$$\int_c^d [\text{Right} - \text{Left}]\, dx$$

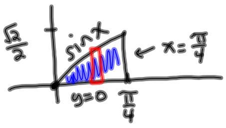


- If we are asked to find the area bounded by the curves $y = f(x)$, $y = g(x)$ where $f(x) \geq g(x)$ for some values of x but $g(x) \geq f(x)$ for other values of x, we must split the integral at each intersection point.

$$S = S_1 + S_2 + S_3$$



EXAMPLE 1: Find the area bounded by $y = \sin x$, $y = 0$, $x = 0$, $x = \frac{\pi}{4}$.



$A = \int (\text{Top} - \text{bottom})\,dx$

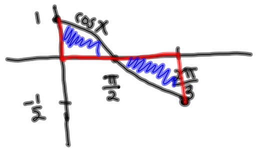
$= \int_0^{\frac{\pi}{4}} (\sin x - 0)\,dx$

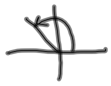
$= -\cos x \Big|_0^{\frac{\pi}{4}}$

$= -\frac{\sqrt{2}}{2} - (-1) = \boxed{-\frac{\sqrt{2}}{2} + 1}$

$= \frac{-\sqrt{2} + 2}{2}$

EXAMPLE 2: Find the area bounded by $y = \cos x$, $y = 0$, $x = 0$, $x = \frac{2\pi}{3}$.





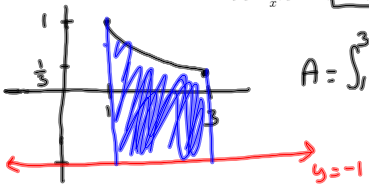
$\int_0^{\frac{\pi}{2}} (\cos x - 0)\,dx + \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} (0 - \cos x)\,dx$

$= \sin x \Big|_0^{\frac{\pi}{2}} - \sin x \Big|_{\frac{\pi}{2}}^{\frac{2\pi}{3}}$

$= 1 - 0 - \left[\frac{\sqrt{3}}{2} - 1\right] = 1 - \frac{\sqrt{3}}{2} + 1$

$= \boxed{2 - \frac{\sqrt{3}}{2}}$

EXAMPLE 3: Find the area bounded by $y = \frac{1}{x}$, $y = -1$, $x = 1$, $x = 3$.



$A = \int_1^3 \left(\frac{1}{x} - (-1)\right) dx$

$= (\ln x + x)\Big|_1^3$

$= \ln 3 + 3 - (\cancel{\ln(1)}^{0} + 1)$

$= \boxed{\ln 3 + 2}$

Last Name, First Name
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ICA #1

2pts

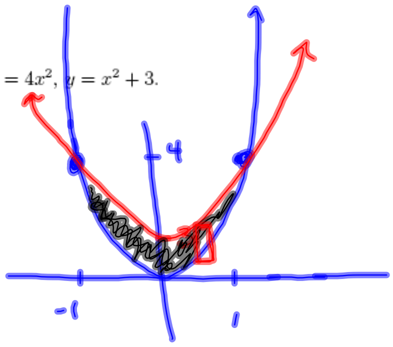
1. $\int_0^{\ln \frac{\pi}{2}} e^x \sin(e^x)\,dx$

2. $\int x\sqrt{x+1}\,dx$

EXAMPLE 4: Find the area bounded by $y = 4x^2$, $y = x^2 + 3$.

1. intersection

$$4x^2 = x^2 + 3$$
$$3x^2 = 3$$
$$x = \pm 1$$



$$A = \int_{-1}^{1} (T - B)\,dx$$

symmetry

$$= 2\int_0^1 \left(x^2 + 3 - 4x^2\right) dx$$
$$= 2\int_0^1 (3 - 3x^2)\,dx$$
$$= 2(3x - x^3)\Big|_0^1 = 4$$

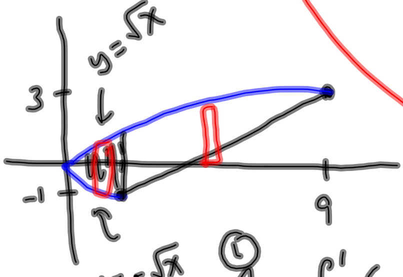
EXAMPLE 5: Find the area bounded by $y^2 = x$, $x - 2y = 3$.

$x = y^2$

$x = 2y + 3$

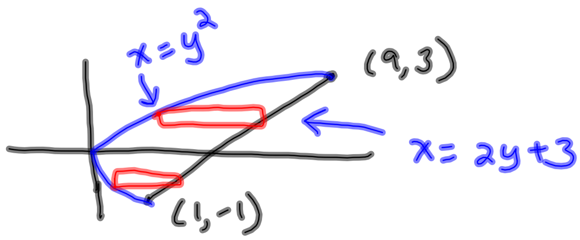
intersection: $y^2 = 2y + 3$ $\quad y = \dfrac{x-3}{2}$

$$y^2 - 2y - 3 = 0$$
$$(y-3)(y+1) = 0$$
$$y = 3,\ x = 9$$
$$y = -1,\ x = 1$$



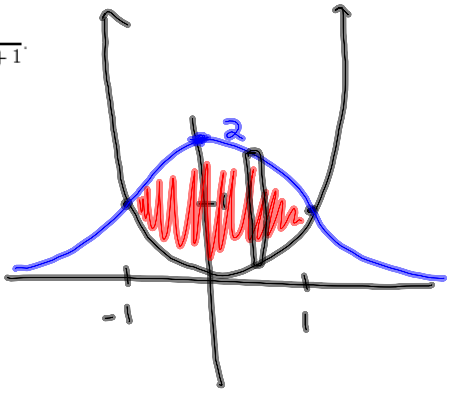
① $A = \int_0^1 \left(\sqrt{x} - (-\sqrt{x})\right) dx + \int_1^9 \sqrt{x} - \left(\frac{x-3}{2}\right) dx$

②



$$A = \int_{-1}^{3} \Big(\underbrace{2y + 3}_{R} - \underbrace{y^2}_{L}\Big) = \left(y^2 + 3y - \frac{y^3}{3}\right)\Big|_{-1}^{3}$$

EXAMPLE 6: Find the area bounded by $y = x^2$, $y = \dfrac{2}{x^2+1}$.



intersect: $x^2 = \dfrac{2}{x^2+1}$

$x^2(x^2+1) = 2$

$x^4 + x^2 = 2$

$x^4 + x^2 - 2 = 0$

$\underbrace{(x^2+2)}_{\text{no solution}}\underbrace{(x^2-1)}_{x=\pm 1} = 0$

$$A = \int_{-1}^{1}\left(\frac{2}{x^2+1} - x^2\right)dx$$

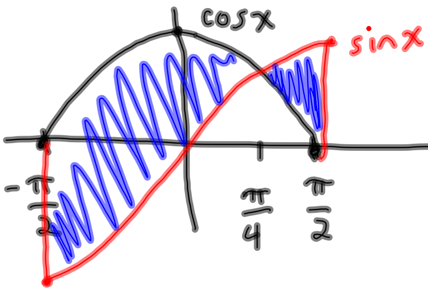
symmetry

$$= 2\int_{0}^{1}\left(\frac{2}{x^2+1} - x^2\right)dx$$

$$= 2\left(2\arctan x - \frac{x^3}{3}\right)\Big|_0^1$$

$$= \boxed{2\left(2\cdot\frac{\pi}{4} - \frac{1}{3}\right)}$$

EXAMPLE 7: Find the area bounded by $y = \sin x$, $y = \cos x$, $x = -\frac{\pi}{2}$, $x = \frac{\pi}{2}$.



$$A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{4}}(\cos x - \sin x)\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(\sin x - \cos x)\,dx$$

$$= (\sin x + \cos x)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{4}} + (-\cos x - \sin x)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

EXAMPLE 8: Find the area bounded by ~~$y = \cos x$, $y = \sin 2x$, $x = 0$, $x = \frac{\pi}{3}$~~

$y = 2\cos(3x)$

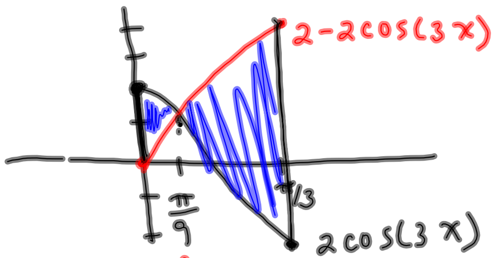
$y = 2 - 2\cos(3x)$

$x = 0$

$x = \frac{\pi}{3}$

$y = 2\cos(3x)$: $x = \frac{\pi}{3}$, $y = -2$; $x = 0$, $y = 2$

$y = 2 - 2\cos(3x)$: $x = \frac{\pi}{3}$, $y = 4$; $x = 0$, $y = 0$



intersect: $2 - 2\cos(3x) = 2\cos(3x)$

$2 = 4\cos(3x) \Rightarrow \frac{1}{2} = \cos(3x)$

recall: $\cos\frac{\pi}{3} = \frac{1}{2}$

$3x = \frac{\pi}{3}$

$x = \frac{\pi}{9}$

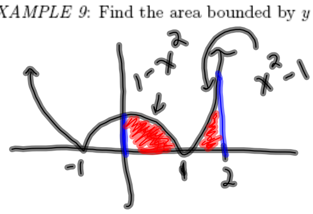
$$A = \int_0^{\frac{\pi}{9}} 2\cos(3x) - \left(2 - 2\cos(3x)\right) + \int_{\frac{\pi}{9}}^{\frac{\pi}{3}} \left(2 - 2\cos(3x) - 2\cos(3x)\right)$$

u-sub $u = 3x$

$$A = \int_0^{\frac{\pi}{9}} \left(4\cos(3x) - 2\right)dx + \int_{\frac{\pi}{9}}^{\frac{\pi}{3}} \left(2 - 4\cos(3x)\right)dx$$

$$= \left(4 \cdot \frac{1}{3}\sin(3x) - 2x\right)\Big|_0^{\frac{\pi}{9}} + \left(2x - 4 \cdot \frac{1}{3}\sin(3x)\right)\Big|_{\frac{\pi}{9}}^{\frac{\pi}{3}}$$

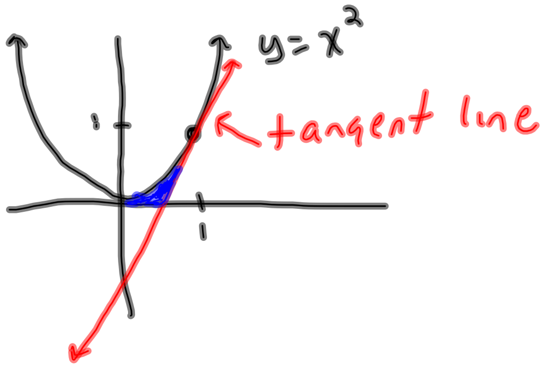
EXAMPLE 9: Find the area bounded by $y = |x^2 - 1|$, $y = 0$, $x = 0$, $x = 2$.

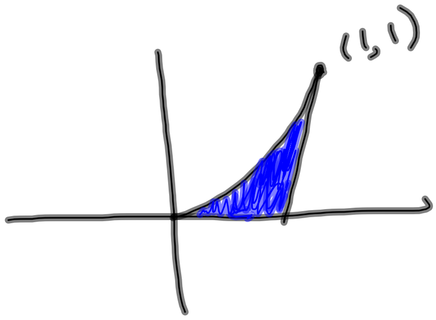


$$A = \int_0^1 (1 - x^2)dx + \int_1^2 (x^2 - 1)dx$$

$$= \left(x - \frac{x^3}{3}\right)\Big|_0^1 + \left(\frac{x^3}{3} - x\right)\Big|_1^2$$

EXAMPLE 10: Find the area of the region bounded by the parabola $y = x^2$, the tangent line to this parabola at $(1, 1)$ and the x-axis.

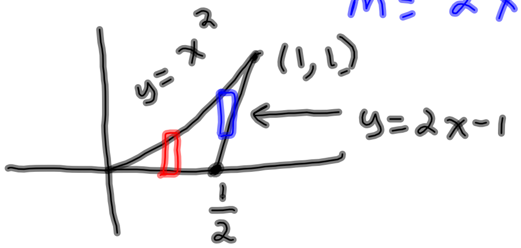




Find tangent line to $y=x^2$ at $(1,1)$

$$m = y'\Big|_{x=1}$$

$$m = 2x\Big|_{x=1} \Rightarrow m = 2$$

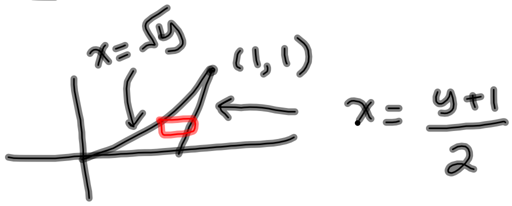


$$y-1 = 2(x-1)$$

$$\boxed{y = 2x-1}$$

① $A = \int_0^{\frac{1}{2}} (x^2 - 0)\,dx + \int_{\frac{1}{2}}^{1} \left(x^2 - (2x-1)\right) dx$

$= \_\_\_$

② 

$$A = \int_0^1 \left(\frac{y+1}{2} - \sqrt{y} \right) dy$$