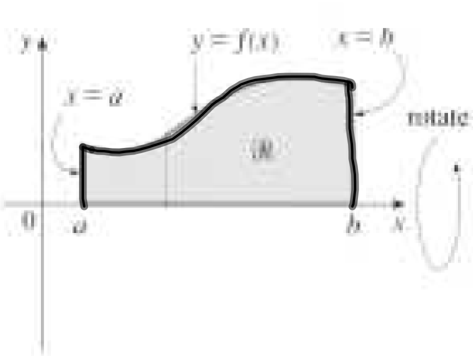
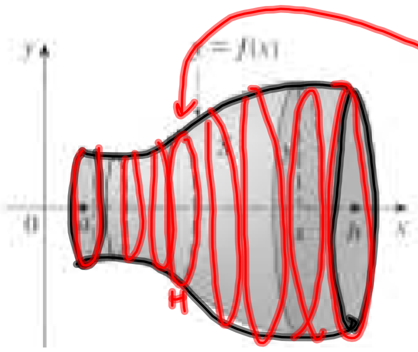
Section 7.2: Volume

1. Disk Method: Use when the cross-section of the solid is in the shape of a disk

- Revolution around the x-axis: $V = \int_a^b \pi \left(f(x)\right)^2 \, dx$



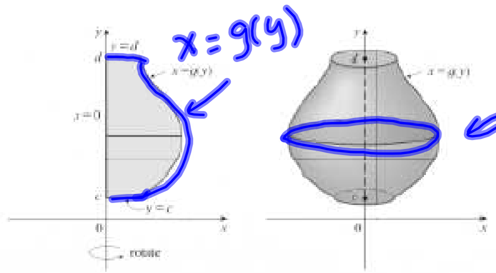


$V = \pi r^2 dx$

$= \pi (f(x))^2 dx$

$V = \int_a^b \pi (f(x))^2 dx$

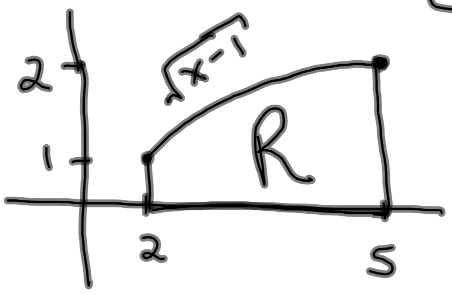
- Revolution around the y-axis: $V = \int_c^d \pi \left(g(y)\right)^2 \, dy$

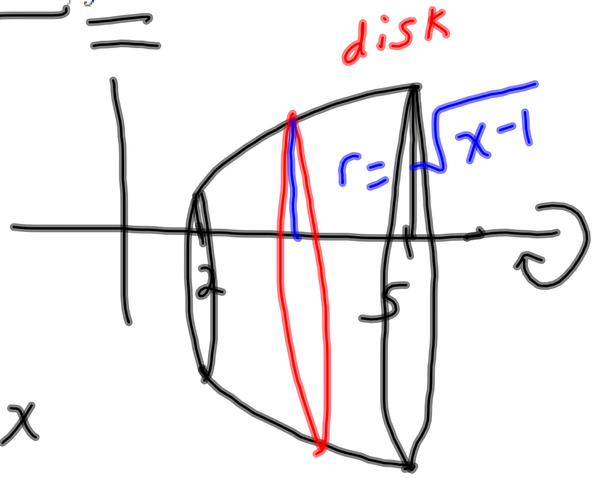


$V = \int \pi r^2 dy$

$= \int_c^d \pi (g(y))^2 dy$

EXAMPLE 1: Find the volume of the solid obtained by rotating the region bounded by $y = \sqrt{x-1}$, $x = 2$, $x = 5$, $y = 0$ about the x axis.



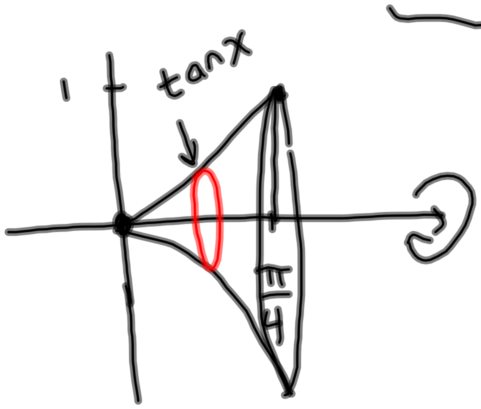


$$V = \int_2^5 \pi\left(\sqrt{x-1}\right)^2 dx$$

$$= \pi \int_2^5 (x-1)\,dx = \pi\left(\frac{x^2}{2} - x\right)\Big|_2^5$$

$$= \_\_\_$$

EXAMPLE 2: Find the volume of the solid obtained by rotating the region bounded by $y = \tan x$, $x = 0$, $x = \frac{\pi}{4}$, $y = 0$ about the x axis.



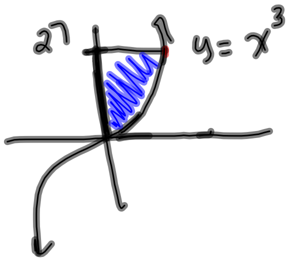
$$V = \int_0^{\frac{\pi}{4}} \pi(\tan x)^2 dx$$

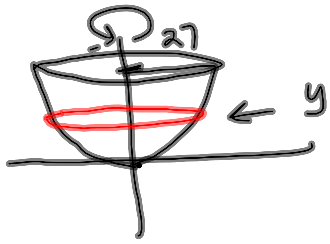
$$= \int_0^{\frac{\pi}{4}} \pi\left(\sec^2 x - 1\right) dx$$

$$= \pi\left(\tan x - x\right)\Big|_0^{\frac{\pi}{4}}$$

$$= \boxed{\pi\left(1 - \frac{\pi}{4}\right)}$$

EXAMPLE 3: Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 27$, $x = 0$ about the y axis.





$\leftarrow y = x^3$

$x = y^{\frac{1}{3}}$

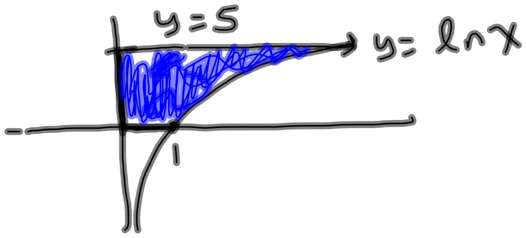
$$V = \int_0^{27} \pi \left(y^{\frac{1}{3}}\right)^2 dy = \int_0^{27} \pi y^{\frac{2}{3}} dy$$

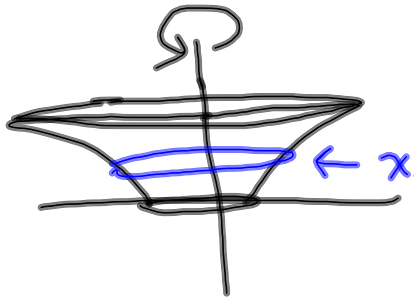
$$= \pi \, \frac{3}{5} y^{\frac{5}{3}} \Big|_0^{27}$$

$$= \frac{3\pi}{5}\left(27^{\frac{5}{3}} - 0\right)$$

$$= \frac{3\pi}{5}\left(3^5\right)$$

EXAMPLE 4: Find the volume of the solid obtained by rotating the region bounded by $y = \ln x$, $x = 0$, $y = 0$, $y = 5$, about the y axis.





$\leftarrow x = e^y$

$$V = \int_0^5 \pi \left(e^y\right)^2 dy$$

$u = 2y$

$du = 2dy$

$$= \pi \int_0^5 e^{2y} dy$$

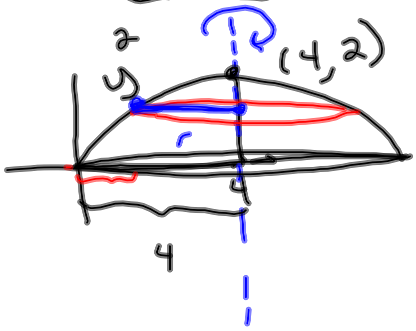
$$= \frac{\pi}{2} \int_0^{10} e^u du = \frac{\pi}{2} e^u \Big|_0^{10}$$

Note: $\int e^{kx} dx = \frac{1}{k} e^{kx}$

$$= \boxed{\frac{\pi}{2}\left(e^{10} - 1\right)}$$

- Revolution around lines:

EXAMPLE 5: Find the volume of the solid obtained by rotating the region bounded by $y = \sqrt{x}$, $y = 0$, $x = 4$ about the line $x = 4$. ← vertical line



disk method

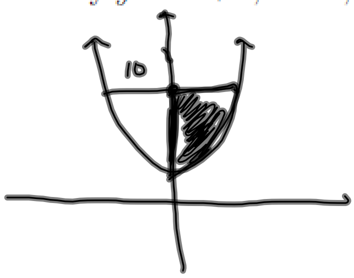
$$V = \int_0^2 \pi r^2 \, dy$$

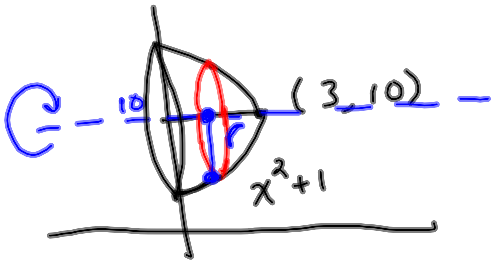
$$= \int_0^2 \pi (4 - y^2)^2 \, dy$$

$$= \pi \int_0^2 (16 - 8y^2 + y^4) \, dy$$

$r = 4 - y^2$

EXAMPLE 6: Find the volume of the solid obtained by rotating the region bounded by $y = x^2 + 1$, $x = 0$, $y = 10$ about the line $y = 10$.





$$V = \int_0^3 \pi r^2 \, dx$$

$r = 10 - (x^2 + 1)$

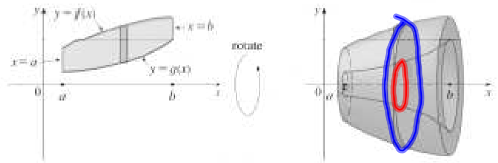
$r = 9 - x^2$

$$= \int_0^3 \pi (9 - x^2)^2 \, dx$$

$$= \pi \int_0^3 (81 - 18x^2 + x^4) \, dx \ldots$$

2. Washer Method: Use when the cross-section of the solid is in the shape of a washer

$$V = \int_a^b \pi \left((f(x))^2 - (g(x))^2 \right) dx$$



outer radius = $f(x)$

inner radius = $g(x)$

$\pi(f(x))^2 - \pi(g(x))^2$

$\pi\left[f(x)^2 - g(x)^2\right]$

EXAMPLE 7: Find the volume of the solid obtained by rotating the region bounded by $y = x^2$, $y = 2x$, about the x axis.

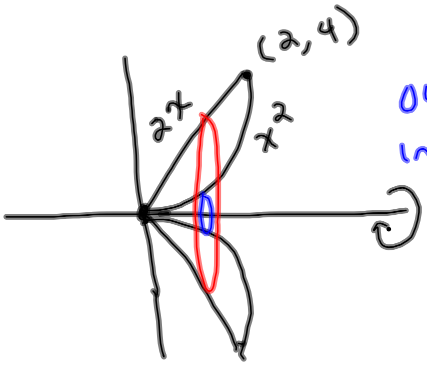
$x^2 = 2x$

$x^2 - 2x = 0$

$x(x-2) = 0$

$x = 0$

$x = 2$

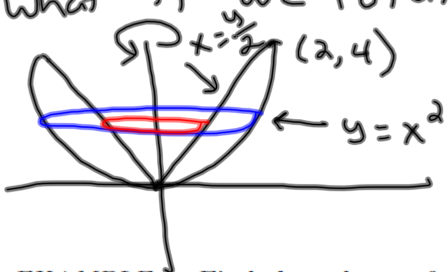


outer radius = $2x$

inner radius = x^2

$$V = \int_0^2 \pi \left((2x)^2 - (x^2)^2 \right) dx$$

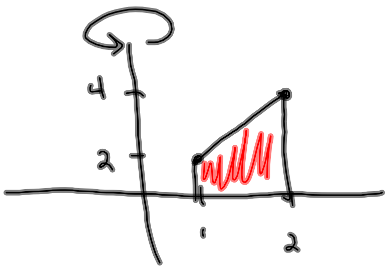
what if we rotate around the y-axis?

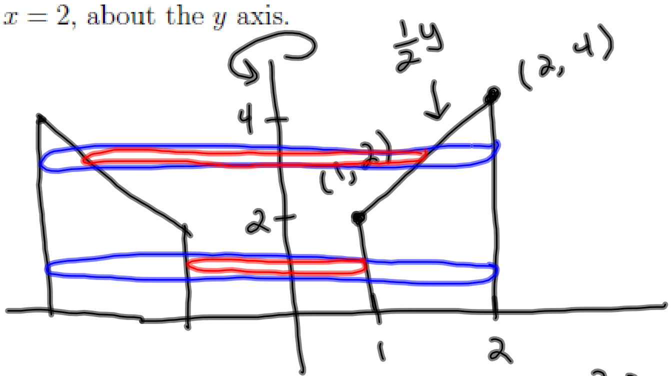


$y = x^2 \Rightarrow x = \sqrt{y}$

$$V = \int_0^4 \pi \left[(\sqrt{y})^2 - \left(\frac{y}{2}\right)^2 \right] dy$$

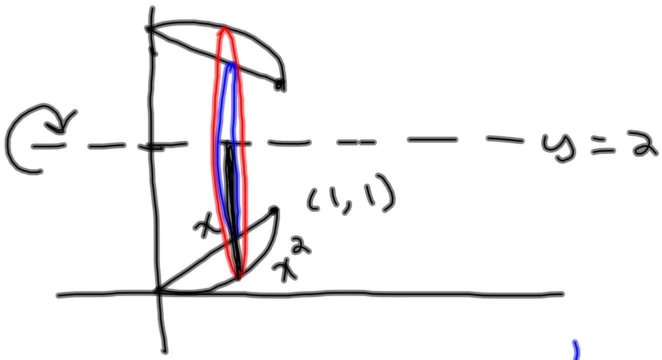
EXAMPLE 8: Find the volume of the solid obtained by rotating the region bounded by $y = 2x$, $y = 0$, $x = 1$, $x = 2$, about the y axis.





$$V = \int_0^2 \pi \left[(2^2 - (1)^2 \right] dy + \int_2^4 \pi \left[2^2 - \left(\frac{1}{2}y\right)^2 \right] dy$$

EXAMPLE 9: Find the volume of the solid obtained by rotating the region bounded by $y = x$, $y = x^2$, about the line $y = 2$.

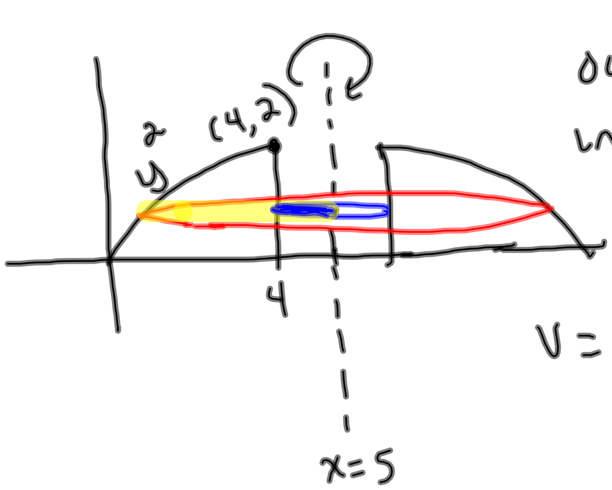


outer radius = $2 - x^2$

inner radius = $2 - x$

$$V = \int_0^1 \pi \left[(2 - x^2)^2 - (2 - x)^2 \right] dx$$

EXAMPLE 10: Find the volume of the solid obtained by rotating the region bounded by $y = \sqrt{x}$, $x = 4$, $y = 0$, about the line $x = 5$.



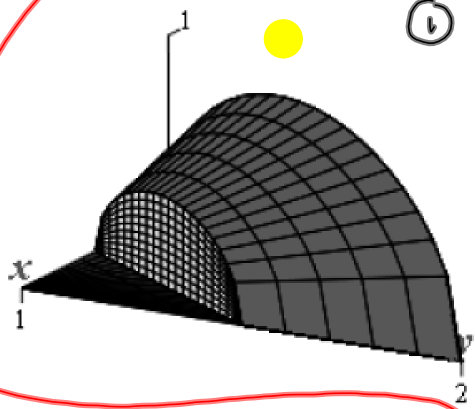
outer radius = $5 - y^2$

inner radius = 1

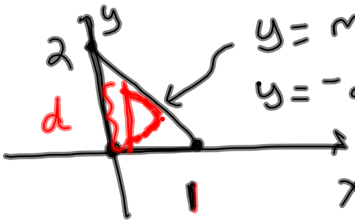
$$V = \int_0^2 \pi \left[(5 - y^2)^2 - 1^2 \right] dy$$

3. The Method of Slicing

EXAMPLE 11: Find the volume of the solid S whose base is the triangular region with vertices $(0,0)$, $(1,0)$ and $(0,2)$. The cross sections of S perpendicular to the x-axis are semi-circles.



① draw base



$y = mx + b$

$y = -2x + 2$

diameter $= d = -2x + 2$

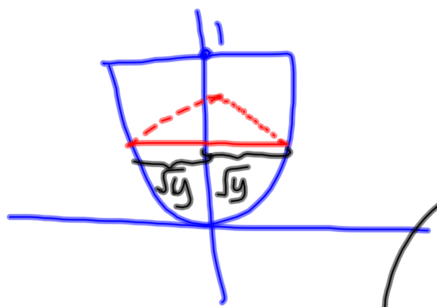
$r = \frac{1}{2} d = -x + 1$

$$V = \int_0^1 \left(A_{\text{semi-circle}}\right) dx = \int_0^1 \frac{1}{2} \pi r^2 \, dx$$

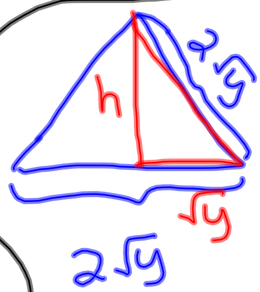
$$= \int_0^1 \frac{1}{2} \pi (-x+1)^2 \, dx$$

EXAMPLE 12: Find the volume of the solid S whose base is the region bounded by the parabola $y = x^2$ and $y = 1$. The cross sections of S perpendicular to the y-axis are equilateral triangles.

dy



$$V = \int_0^1 (A_\Delta)(dy)$$



$$b = 2\sqrt{y}$$

$$4y = h^2 + y$$

$$h^2 = 3y$$

$$h = \sqrt{3y}$$

$$A_\Delta = \frac{1}{2} b h$$

$$= \frac{1}{2} 2\sqrt{y}\sqrt{3y}$$

$$A_\Delta = \sqrt{3}\, y$$

$$\boxed{V = \int_0^1 \sqrt{3}\, y \, dy = \frac{\sqrt{3}}{2}}$$

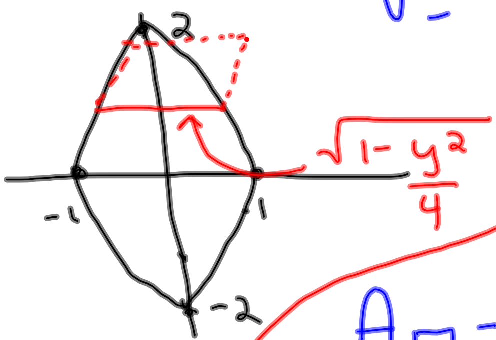
EXAMPLE 13: Find the volume of the solid S whose base is the ellipse $x^2 + \frac{y^2}{4} = 1$. The cross sections of S perpendicular to the y-axis are squares.

$\to dy$

$$x^2 + \frac{y^2}{4} = 1$$

$$x^2 = 1 - \frac{y^2}{4}$$

$$x = \sqrt{1 - \frac{y^2}{4}}$$



$$V = \int_{-2}^{2} A_{\square}\, dy$$

$$A_{\square} = \left(2\sqrt{1 - \frac{y^2}{4}}\right)^2$$

$$A_{\square} = 4\left(1 - \frac{y^2}{4}\right)$$

$$V = \int_{-2}^{2} 4\left(1 - \frac{y^2}{4}\right) dy$$

$$= \int_{-2}^{2} (4 - y^2)\, dy$$

~~*EXAMPLE 14*: Find the volume of the cap of a sphere with radius 4~~ and ~~height 1.~~

OMIT