

Section 7.4: Work

Definition If an object is moving along a straight path, then the Force F acting on the object is the object's mass times the acceleration due to gravity. Intuitively, we can think of force as the push or pull on an object. In general, if an object with mass m moves along a straight path with position function $s(t)$, then the force is given by

$$F = m \frac{d^2 s}{dt^2}$$

In the case where the force is constant, then the Work W done in moving an object a distance d meters is given by

$$W = Fd$$

1. How much work is done in raising a 60 kg barbell from the floor to a height of 2 meters?

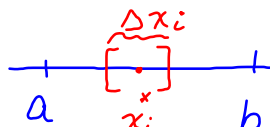
$$\begin{aligned} w &= F \cdot d \\ &= (60 \text{ kg}) \left(9.8 \frac{\text{m}}{\text{s}^2} \right) (2 \text{ m}) \\ &= 1176 \text{ Joules} \end{aligned}$$

2. How much work is done in lifting a 50 pound weight from a height of 6 inches to a height of 18 inches?

$$\begin{aligned} W &= F \cdot d \\ &= (50 \text{ lbs}) (12 \text{ inches}) \\ &= (50 \text{ lbs}) (1 \text{ foot}) \\ &: 50 \text{ ft lbs} \end{aligned}$$

What if the force is not constant? In this case, finding the work requires integration. We will derive the procedure here:

Suppose we are moving an object from $x=a$ to $x=b$ via a variable force $f(x)$.

 For this small interval, the force is almost constant, so

$$W_i: \underbrace{f(x_i^*)}_{\text{force}} \underbrace{(\Delta x_i)}_{\text{distance}}$$

$$W_{\text{total}} \approx \sum_{i=1}^n f(x_i^*) \Delta x_i$$

thus
$$W = \int_a^b f(x) dx$$

3. When a particle is at a distance x meters from the origin, a force of $f(x) = 2x^2 + 1$ Newtons acts on the object, thus causing the object to move horizontally along the x -axis. How much work is done in moving the object from $x = 1$ to $x = 2$?

$$W = \int_1^2 (2x^2 + 1) dx$$

$$= \left(\frac{2x^3}{3} + x \right) \Big|_1^2$$

$$= \frac{16}{3} + 2 - \left(\frac{2}{3} + 1 \right)$$

$$= \frac{17}{3} \text{ Joules}$$

4. A spring has a natural length of 1 m. If a 25-N force is required to keep it stretched to a length of 3 m, how much work is done in stretching the spring from 2 m to 5 m? Here we will use Hooke's Law which states the force required to maintain a spring stretched x units beyond its natural length is given by $f(x) = kx$.



Keeping it stretched
2 m beyond its
natural length
under a force of
25 N $\rightarrow f(2) = 25$

force function $f(x) = kx$

$2k = 25$
 $k = \frac{25 \text{ N}}{2 \text{ m}}$ subtract

$\therefore f(x) = \frac{25}{2} x$

$W = \int_1^4 \frac{25}{2} x dx$

$= \frac{25}{4} x^2 \Big|_1^4 = \frac{25}{4} (15)$

$\rightarrow \text{integrate} = \boxed{\frac{375 \text{ J}}{4}}$

5. If the work required to stretch a spring 1 foot beyond its natural length is 12 foot pounds, how much work is needed to stretch it 9 inches beyond its natural length?

given: $\int_0^1 kx dx = 12$

$k \frac{x^2}{2} \Big|_0^1 = 12$

$k = 24$ $9 \text{ in} = \frac{9}{12} = \frac{3}{4} \text{ ft}$

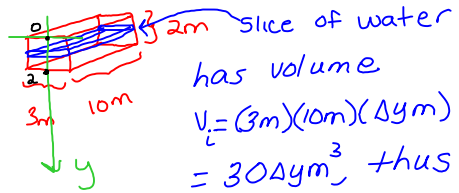
$f(x) = 24x$

$W = \int_0^{3/4} 24x dx$

$= 12x^2 \Big|_0^{3/4}$

$= \frac{27}{4} \text{ ft lbs}$

6. A rectangular tank 10 m long, 3 m wide and 2 m deep is filled with water. Find the work required to pump all the water to the top of the tank. How much work is done in pumping half the water to the top of the tank?



slice of water has volume
 $V_i = (3m)(10m)(\Delta y m)$
 $= 30\Delta y m^3$, thus

weighs $F_i = 30\rho g \Delta y$ N
 where $\rho g = 9800$ N/m³
 and it must be lifted a distance y m, thus

$$W_i = (30\rho g \Delta y)(y) Nm$$

① work to pump all water

$$W = \int_0^2 30\rho g y dy$$

$$= 15\rho g y^2 \Big|_0^2$$

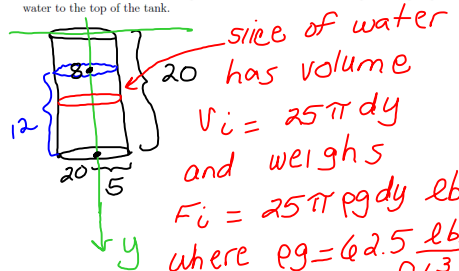
$$= \boxed{60\rho g N}$$

② work to pump half water

$$W = \int_0^1 30\rho g y dy$$

$$= 15\rho g N$$

7. A tank is in the shape of an upright cylinder with radius 5 feet and height 20 feet. The tank is full of water to a depth of 12 feet. Find the work required to pump the water to the top of the tank.



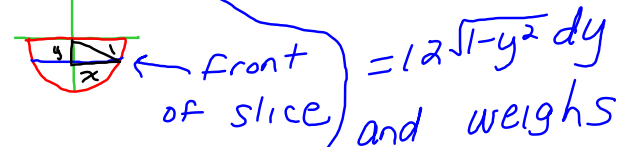
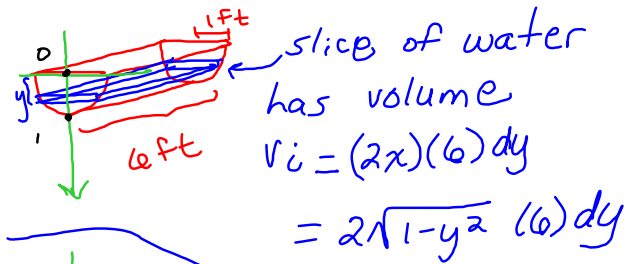
slice of water has volume
 $V_i = 25\pi dy$
 and weighs
 $F_i = 25\pi\rho g dy$ lbs
 where $\rho g = 62.5 \frac{\text{lbs}}{\text{ft}^3}$

and must be lifted

$$d = y \text{ feet}$$

$$W = \int_8^{20} 25\pi\rho g y dy = 4200\pi\rho g \text{ ft lbs}$$

8. A trough is in the shape of a half drum (half a cylinder lying on its side). The length of the tank is 6 feet and the radius is 1 foot. Assuming it is full of water, find the work done in pumping the water to the top of the tank.



$$x^2 + y^2 = 1$$

$$x^2 = 1 - y^2$$

$$x = \sqrt{1 - y^2}$$

$$F_i = 12\rho g \sqrt{1 - y^2} dy$$

where $\rho g = 62.5 \frac{\text{lb}}{\text{ft}^3}$

and must be moved a distance y m

$$W = \int_0^1 12\rho g y \sqrt{1 - y^2} dy$$

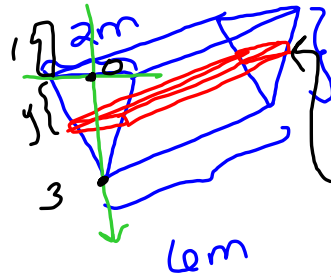
u-sub $u = 1 - y^2$
 $du = -2y dy$

$$= \int_1^0 \frac{12\rho g}{-2} u^{\frac{1}{2}} du$$

$$= 6\rho g \frac{2}{3} u^{\frac{3}{2}} \Big|_0^1$$

$$= 4\rho g \text{ ft lbs}$$

9. A triangular trough has a length of 6 m, a distance of 2 m across the top and a height of 3 m. Assuming it is full of water, find the work done in pumping the water through a spout located at the top of the trough with length 1 m.



spout adds to distance

$$V_i = (2x)(6)(dy)$$

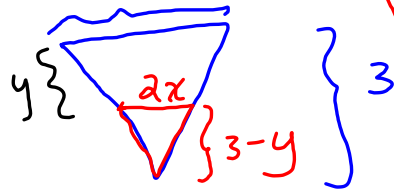
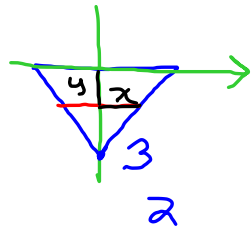
$$= \frac{2}{3}(3-y)(6)dy$$

$$= 4(3-y)dy$$

$$F_i = 4\rho g(3-y)dy$$

$$\rho g = 9800 \frac{N}{m^3}$$

$$\text{distance} = y + 1$$



sim triangles

$$\frac{2x}{3-y} = \frac{2}{3}$$

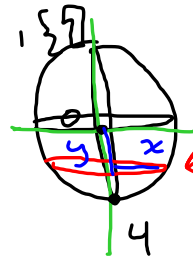
$$2x = \frac{2}{3}(3-y)$$

$$W = \int_0^3 4\rho g(3-y)(y+1)dy$$

$$= 4\rho g \int_0^3 (3y+3-y^2-y)dy$$

$$= \text{---} \text{ J}$$

10. A spherical tank of radius 4 m is half full of a liquid that has density of 900 kg per cubic meter. Find the work done in pumping the water through a spout located at the top of the trough with length 1 m.



$$V_{\text{slice}} = \pi x^2 dy$$

where $x^2 + y^2 = 16$

$$V_i = \pi (16 - y^2) dy$$

$$F_i = \pi \rho g (16 - y^2) dy$$

where $\rho g = (900)(9.8) \text{ N/m}^3$

slice must travel $d = y + 5 \text{ m}$

$$\text{thus } W = \int_0^4 \pi \rho g (16 - y^2) (y + 5) dy$$

$$= \pi \rho g \int_0^4 (16y + 80 - y^3 - 5y^2) dy$$

$$= \text{--- J}$$

11. A 200 pound cable is 100 feet long and hangs vertically from the top of a tall building. How much work is required to pull the cable to the top of the building? How much work is done in pulling half the cable to the top of the building?

$$\text{Rope weighs } \frac{200 \text{ lbs}}{100 \text{ ft}} = 2 \frac{\text{lbs}}{\text{foot}}$$

The work done in pulling the entire rope is

$$\begin{aligned} W &= \int_0^{100} 2y \, dy \\ &= \left. \frac{y^2}{1} \right|_0^{100} \\ &= (100)^2 \text{ ft lbs} \end{aligned}$$

The work done in pulling half the rope is

$$\begin{aligned} W &= \int_0^{50} y \, dy + (50)(2)(50) \\ &= \left. \frac{y^2}{2} \right|_0^{50} + 5000 \\ &= 6250 \text{ ft lbs} \end{aligned}$$