Integration by parts: $$\int u\,dv = uv - \int v\,du$$

Proof: Recall by the product rule that $(uv)' = u'v + uv'$. Integrate both sides:
$\int (uv)' = \int u'v + \int uv'$. Thus $uv = \int u'v + \int uv'$, hence $\int uv' = uv - \int u'v$

must be degree one.

1. Integrals of the form $\int x^n e^{kx} dx$ ← If not, u-sub first where u= power on e.

a.) $\int xe^{2x}\,dx$

$u = x^n$
$dv = e^{kx}\,dx$

Here, $u = x$ $\quad dv = e^{2x}\,dx$

$du = dx \quad v = \int e^{2x}\,dx = \frac{1}{2}e^{2x}$

$$\int \underbrace{x}_{u}\underbrace{e^{2x}\,dx}_{dv} = uv - \int v\,du$$

$$= (x)\left(\tfrac{1}{2}e^{2x}\right) - \int \tfrac{1}{2}e^{2x}\,dx$$

$$= \boxed{\tfrac{1}{2}xe^{2x} - \tfrac{1}{4}e^{2x} + C}$$

b.) $\int_0^1 3xe^{-x}\,dx$

Find $\int 3xe^{-x}\,dx$ $\quad u = 3x \quad dv = e^{-x}\,dx$

$du = 3\,dx \quad v = -e^{-x}$

$$\int \underbrace{3x}_{u}\underbrace{e^{-x}\,dx}_{dv} = uv - \int v\,du$$

$$= \underbrace{(3x)}_{u}\underbrace{(-e^{-x})}_{v} - \int \underbrace{(-e^{-x})}_{v}\underbrace{(3\,dx)}_{du}$$

$$= -3xe^{-x} + \int 3e^{-x}\,dx = -3xe^{-x} - 3e^{-x}$$

$$\int_0^1 3xe^{-x}\,dx = \left(-3xe^{-x} - 3e^{-x}\right)\Big|_0^1$$

$$= -3e^{-1} - 3e^{-1} - (0-3) = \boxed{-6e^{-1} + 3 \text{ or } \frac{-6}{e} + 3}$$

c.) $\int x^5 e^{x^3}\,dx$ ← t - sub

$t = x^3$

$dt = 3x^2\,dx \Rightarrow \frac{dt}{3x^2} = dx$

$\int x^5 e^t \frac{dt}{3x^2}$

parts: $u = t \quad dv = e^t dt$

$du = dt \quad v = e^t$

$\frac{1}{3}\int x^3 e^t\,dt = \frac{1}{3}\int t e^t\,dt$

$= \frac{1}{3}\left(uv - \int v\,du\right) = \frac{1}{3}\left[te^t - \int e^t dt\right]$

$= \frac{1}{3}\left[te^t - e^t\right] + C$

since $t = x^3$,

$= \frac{1}{3}\left[x^3 e^{x^3} - e^{x^3}\right] + C$

2. Integrals of the form $\int x^n \sin(kx)\,dx$ or $\int x^n \cos(kx)\,dx$

must be degree one.

If not do substitution first

a.) $\int (x+1)\sin(5x)\,dx$

$t = $ angle $\quad u = x^n$

$dv = $ what's left

$u = x+1 \qquad dv = \sin(5x)\,dx$

$du = dx \qquad v = -\frac{1}{5}\cos(5x)$

integrate picks up another $\frac{1}{5}$

$\int \underbrace{(x+1)}_{u}\underbrace{\sin(5x)\,dx}_{dv} = uv - \int v\,du$

$= \underbrace{(x+1)}_{u}\underbrace{\left(-\frac{1}{5}\cos(5x)\right)}_{v} - \int \underbrace{-\frac{1}{5}\cos(5x)}_{v}\,\underbrace{dx}_{du}$

$= -\frac{1}{5}(x+1)\cos(5x) + \frac{1}{25}\sin(5x) + C$

b.) $\int x^2 \cos(x)\,dx$

$u = x^2 \quad dv = \cos(x)\,dx$

$du = 2x\,dx \quad v = \sin x$

$$\int x^2\cos(x)\,dx = uv - \int v\,du$$

$$= x^2\sin x - \int (\sin x)(2x\,dx)$$

$$= x^2 \sin x - \boxed{\int 2x\sin x\,dx} \rightarrow uv - \int v\,du$$

parts:

$u = 2x,\ dv = \sin x\,dx$

$du = 2\,dx,\ v = -\cos x$

$-2x\cos x - \int -2\cos x\,dx$

$-2x\cos x + 2\sin x$

$$= x^2\sin x - \left[-2x\cos x + 2\sin x\right] + C$$

$$= x^2\sin x + 2x\cos x - 2\sin x + C$$

3. Integrals of the form $\int x^n \ln x\,dx$ $\quad u = \ln x \quad dv = x^n\,dx$

a.) $\int x^9 \ln x\,dx$

$u = \ln x, \quad dv = x^9\,dx$

$du = \frac{1}{x}\,dx, \quad v = \frac{x^{10}}{10}$

$$\int x^9 \ln x\,dx = uv - \int v\,du$$

$$= (\ln x)\left(\frac{x^{10}}{10}\right) - \int \frac{x^{10}}{10}\,\frac{1}{x}\,dx$$

$$= \frac{x^{10}}{10}\ln x - \frac{1}{10}\int x^9\,dx$$

$$= \boxed{\frac{x^{10}}{10}\ln x - \frac{1}{100}x^{10} + C}$$

b.) $\int_1^4 \ln \sqrt{x}\, dx = \int_1^4 \ln x^{\frac{1}{2}}\, dx = \frac{1}{2}\int_1^4 \ln x\, dx$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x}\, dx \qquad v = x$

$$\int \ln x\, dx = uv - \int v\, du$$
$$= x\ln x - \int x \cdot \frac{1}{x}\, dx$$
$$= x\ln x - \int dx$$
$$= x\ln x - x$$

$$= \frac{1}{2}\left[x\ln x - x\right]\Big|_1^4$$
$$= \frac{1}{2}\left[4\ln 4 - 4 - (0-1)\right]$$
$$= \frac{1}{2}\left[4\ln 4 - 3\right]$$
$$= 2\ln 4 - \frac{3}{2}$$
$$= \ln(16) - \frac{3}{2}$$

4. Integrals involving Inverse Trig Functions:

a.) $\int \arcsin x\, dx$

$u =$ inverse trig

$dv = dx$

$u = \arcsin x \qquad dv = dx$

$du = \frac{1}{\sqrt{1-x^2}}\, dx \qquad v = x$

$$\int \arcsin x\, dx = uv - \int v\, du$$
$$= x\arcsin x - \underbrace{\int x \frac{1}{\sqrt{1-x^2}}\, dx}$$

u-sub

$u = 1 - x^2$

$du = -2x\, dx$

$$-\frac{1}{2}\int \frac{du}{\sqrt{u}} = -\frac{1}{2}\int u^{-\frac{1}{2}}\, du$$
$$= -\frac{1}{2}\frac{u^{\frac{1}{2}}}{\frac{1}{2}}$$
$$= -\sqrt{1-x^2}$$

$$= \boxed{x\arcsin x + \sqrt{1-x^2} + C}$$

b.) change to $\int_0^1 x\arctan x\,dx$

$u = \arctan x \qquad dv = x\,dx$

$du = \dfrac{dx}{1+x^2} \qquad v = \dfrac{x^2}{2}$

$$\int x\arctan x = uv - \int v\,du$$

$$= \frac{x^2}{2}\arctan x - \int \frac{x^2}{2}\,\frac{dx}{1+x^2}$$

$$= \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx$$

$$= \frac{x^2}{2}\arctan x - \frac{1}{2}\int \frac{x^2+1-1}{1+x^2}\,dx$$

$$= \frac{x^2}{2}\arctan x - \frac{1}{2}\left[\int \underbrace{\frac{x^2+1}{x^2+1}\,dx}_{dx} - \frac{1}{x^2+1}\,dx\right]$$

$$= \frac{x^2}{2}\arctan x - \frac{1}{2}\left[\, x - \arctan x \,\right]$$

$$\int_0^1 x\arctan x\,dx = \left(\frac{x^2}{2}\arctan x - \frac{1}{2}x + \frac{1}{2}\arctan x\right)\Big|_0^1$$

$$= \boxed{\frac{1}{2}\frac{\pi}{4} - \frac{1}{2} + \frac{1}{2}\frac{\pi}{4}}$$

$u = e^{2x}, \quad dv = \cos(4x)\,dx$

5. Integrals involving loops:

$du = 2e^{2x}\,dx, \quad v = \frac{1}{4}\sin(4x)$

a.) $\int e^{2x}\cos(4x)\,dx$

$$\int e^{2x}\cos 4x\,dx = uv - \int v\,du$$

$$= \frac{1}{4}e^{2x}\sin(4x) - \int \frac{1}{2}e^{2x}\sin(4x)\,dx$$

parts

$u = e^{2x}, \quad dv = \sin(4x)\,dx$

$du = 2e^{2x}\,dx, \quad v = -\frac{1}{4}\cos(4x)$

$$= \frac{1}{4}e^{2x}\sin(4x) - \frac{1}{2}\int e^{2x}\sin(4x)\,dx$$

$$= \frac{1}{4}e^{2x}\sin(4x) - \frac{1}{2}\left[\underbrace{e^{2x}}_{u}\underbrace{\left(-\frac{1}{4}\cos(4x)\right)}_{v} - \int \underbrace{-\frac{1}{4}\cos(4x)}_{v}\,\underbrace{2e^{2x}\,dx}_{du}\right]$$

$$= \frac{1}{4}e^{2x}\sin(4x) - \frac{1}{2}\left[-\frac{1}{4}e^{2x}\cos(4x) + \frac{1}{2}\int e^{2x}\cos(4x)\,dx\right]$$

$$= \frac{1}{4}e^{2x}\sin(4x) + \frac{1}{8}e^{2x}\cos(4x)\,\underbrace{-\frac{1}{4}\int e^{2x}\cos(4x)\,dx}_{\text{add to other side}}$$

$$\frac{5}{4}\int e^{2x}\cos(4x)\,dx = \frac{1}{4}e^{2x}\sin(4x) + \frac{1}{8}e^{2x}\cos(4x)$$

$$\boxed{\int e^{2x}\cos(4x)\,dx = \frac{4}{5}\left[\frac{1}{4}e^{2x}\sin(4x) + \frac{1}{8}e^{2x}\cos(4x)\right] + C}$$

b.) $\int \sec^3 x\,dx$

First find $\int \sec x\,dx$

$$\int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x}\right) dx$$

$$= \int \frac{\overbrace{\sec^2 x + \sec x \tan x\,dx}^{du}}{\underbrace{\sec x + \tan x}_{u}}$$

$u = \sec x + \tan x \Rightarrow du = (\sec x \tan x + \sec^2 x)\,dx$

$$= \int \frac{du}{u} = \ln|u| + C = \boxed{\ln|\sec x + \tan x| + C}$$

Now find $\int \sec^3 x\,dx = \int \sec x \sec^2 x\,dx$

parts:

$u = \sec x$, $dv = \sec^2 x\,dx$

$du = \sec x \tan x\,dx$, $v = \tan x$

$$\int \sec^3 x\,dx = uv - \int v\,du$$

$$= \sec x \tan x - \int \sec x \tan^2 x\,dx$$

$\tan^2 x \to \sec^2 x - 1$

$$= \sec x \tan x - \int \sec x(\sec^2 x - 1)\,dx$$

$$= \sec x \tan x - \int (\sec^3 x - \sec x)\,dx$$

$$= \sec x \tan x - \underbrace{\int \sec^3 x\,dx}_{\text{add to other side}} + \boxed{\int \sec x\,dx}$$

just did this

$$2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x|$$

$$\int \sec^3 x\,dx = \frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) + C$$