Section 8.9: Improper Integrals

Case I: Integrals where one (or both) of the limits is infinite: Your goal is to determine whether the improper integral converges (finite value) or diverges (infinite value).

- $\displaystyle\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$
- $\displaystyle\int_{-\infty}^a f(x)\,dx = \lim_{t\to-\infty}\int_t^a f(x)\,dx$
- $\displaystyle\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$, then try to evaluate both integrals.

1. $\displaystyle\int_2^\infty \frac{1}{\sqrt{x+3}}\,dx$

2. $\displaystyle\int_2^\infty \frac{dx}{(x+3)^2}$

3. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+4}$

4. $\displaystyle\int_{e}^{\infty} \frac{dx}{x(\ln x)^2}$

Case II: Integrals where there is a discontinuity on the interval $[a,b]$:

- Suppose $f(x)$ is discontinuous at $x=a$: Then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

- Suppose $f(x)$ is discontinuous at $x=b$: Then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

- If $f(x)$ is discontinuous at some c where $a<c<b$, then

$\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, then try to evaluate both integrals.

5. $\displaystyle\int_1^9 \frac{1}{\sqrt[3]{x-9}}\,dx$

6. $\displaystyle\int_{-1}^{8} \frac{1}{\sqrt[3]{x}}\, dx$

7. $\displaystyle\int_{-1}^{2} \frac{1}{x^4}\, dx$

Comparison Theorem for Improper Integrals:
Suppose $f(x)$ and $g(x)$ are continuous, positive functions on the interval $[a, \infty)$. Also, suppose that $f(x) \geq g(x)$ on the interval $[a, \infty)$. Then:

(i) If $\int_a^\infty f(x)\,dx$ converges, so does $\int_a^\infty g(x)\,dx$.

(Note: If $\int_a^\infty f(x)\,dx$ diverges, no conclusion can be drawn about $\int_a^\infty g(x)\,dx$).

(ii) If $\int_a^\infty g(x)\,dx$ diverges, so does $\int_a^\infty f(x)\,dx$.

(Note: If $\int_a^\infty g(x)\,dx$ converges, no conclusion can be drawn about $\int_a^\infty f(x)\,dx$).

Note: The way you choose the comparison function: You take the largest part of the numerator over the largest part of the denominator on the interval $[a, \infty)$. Once you find the comparison function, you *must* determine the direction of the inequality, then integrate the comparison function and draw the correct conclusion.

8. Determine whether the following integrals converge or diverge:

   a.) $\displaystyle\int_1^\infty \frac{1}{x + e^{2x}}\,dx$

b.) $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^4}\,dx$

c.) $\displaystyle\int_0^{\pi/2} \frac{dx}{x \sin x}$