Section 9.5: Moments and Centers of Gravity

The purpose of this section is to find where a system balances. We will start with the most simple of situations, one dimensional, and work our way to two dimensional. In Calculus III, you will investigate 3-D.

Case I If we have a system of n particles with masses $m_1, m_2, ..., m_n$ located at the points $x_1, x_2, ..., x_n$ on the x axis, then

$$\overline{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}$$

EXAMPLE 1: Find the center of mass of a system of 4 objects with masses 10g, 45g, 32g and 24 g that are located at the points $x = -4, 1, 3$ and 8 along the x axis, respectively.

Case II If we have a system of n particles with masses $m_1, m_2, ..., m_n$ located at the points $(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)$ in the x-y plane.

- The **moment of the system about the y axis** is $M_y = \sum_{i=1}^{n} m_i x_i$. This measures the tendency of the system to rotate about the y axis.
- The **moment of the system about the x axis** is $M_x = \sum_{i=1}^{n} m_i y_i$. This measures the tendency of the system to rotate about the x axis.
- The **center of mass** is $(\overline{x}, \overline{y})$ where

$$\overline{x} = \frac{M_y}{\sum_{i=1}^{n} m_i} \qquad \text{and} \qquad \overline{y} = \frac{M_x}{\sum_{i=1}^{n} m_i}$$

EXAMPLE 2: The masses $m_1 = 2$, $m_2 = 3$, and $m_3 = 1$ are located at the points $(5, 1)$, $(3, -2)$, and $(-2, 4)$, respectively. Find M_x, M_y and the center of mass.

Case III Now we have a function $y = f(x)$ with uniform density ρ on the interval $[a, b]$.

- The moment about the y-axis is:

$$M_y = \rho \int_a^b x f(x)\, dx$$

- The moment about the x-axis is:

$$M_x = \rho \int_a^b \frac{1}{2}(f(x))^2\, dx$$

- The x coordinate of the centroid is

$$\overline{x} = \frac{1}{A} \int_a^b x f(x)\, dx$$

where A is the area of the region.

- The y coordinate of the centroid is

$$\overline{y} = \frac{1}{A} \int_a^b \frac{1}{2}[f(x)]^2\, dx$$

where A is the area of the region.

EXAMPLE 3: Find the centroid of the region bounded by the curves $y = 1 - x^2$ and $y = 0$

EXAMPLE 4: Find the centroid of the region bounded by the curves $y = \sin x$, $y = 0$, $x = 0$, $x = \dfrac{\pi}{2}$

EXAMPLE 5: Find the centroid of the quarter circle $x^2 + y^2 = 4$, $0 \le x \le 2$.

Case IV Now we have the region with density ρ bounded by
$y = f(x)$ and $y = g(x)$, where $f(x) \geq g(x)$ over the interval $[a, b]$.

- The moment about the y-axis is:

$$M_y = \rho \int_a^b x(f(x) - g(x))\, dx$$

- The moment about the x-axis is:

$$M_x = \rho \int_a^b \frac{1}{2}(f(x))^2 - (g(x))^2)\, dx$$

- The x coordinate of the centroid is

$$\overline{x} = \frac{1}{A} \int_a^b x(f(x) - g(x))\, dx$$

where A is the area of the region.

- The y coordinate of the centroid is

$$\overline{y} = \frac{1}{A} \int_a^b \frac{1}{2}([f(x)]^2 - [g(x)]^2)\, dx$$

where A is the area of the region.
EXAMPLE 6: Find the centroid of the region bounded by $y = x^2$ and $y = \sqrt{x}$.