SERIES

courtesy of Amy Austin

Def: Let \( \{a_n\} = \{a_1, a_2, a_3, \ldots, \ldots\} \) be a sequence. We define the infinite series to be \( \sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots + \ldots \). In other words, a series is the sum of a sequence. The main focus of chapter 10 is to determine when the sum is finite.

Def: Let \( \sum a_n \) be a series. We will construct the sequence of partial sums \( \{s_n\} = \{s_1, s_2, s_3, \ldots, \ldots\} \) as follows:

\[
\begin{align*}
    s_1 &= a_1 \\
    s_2 &= a_1 + a_2 \\
    s_3 &= a_1 + a_2 + a_3
\end{align*}
\]

Therefore a general formula for \( s_n \) is

\[
s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n.
\]

If \( \lim_{n \to \infty} s_n = s \), where \( s \) is finite, then we say the series \( \sum a_n \) converges and it’s sum is \( s \). If \( \lim_{n \to \infty} s_n \) is infinite or does not exist, then we say the series \( \sum a_n \) diverges.

Test for Convergence

Below are the various tests to determine whether a particular series converges or diverges.

1. The Test for Divergence: If \( \lim_{n \to \infty} a_n \neq 0 \), then \( \sum a_n \) diverges. NOTE: The converse is not necessarily true: If \( \lim_{n \to \infty} a_n = 0 \), then the series \( \sum a_n \) does not necessarily converge. Therefore if you find that \( \lim_{n \to \infty} a_n = 0 \), then the divergence test fails. For example the series \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges, however the TERMS \( \{\frac{1}{n}\} \) do go to zero-just not fast enough to get a finite SUM.

2. Geometric series: The geometric series \( \sum_{n=1}^{\infty} ar^{n-1} \) converges if \( |r| < 1 \) and diverges if \( |r| \geq 1 \). If \( |r| < 1 \), then the sum is \( \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r} \).
3. **The Integral Test:** If \( f(x) \) is a positive, continuous, decreasing function on \([1, \infty]\), and \( a_n = f(n) \). Then:

a.) If \( \int_1^\infty f(x) \, dx \) is convergent, then \( \sum_{n=1}^{\infty} a_n \) converges.

b.) If \( \int_1^\infty f(x) \, dx \) diverges, then \( \sum_{n=1}^{\infty} a_n \) diverges.

4. The **p-series** \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) is convergent if \( p > 1 \) and divergent if \( p \leq 1 \).

5. **The Comparison Test:** (Use this test if the series is a series of positive terms, and the series is comparable to a p-series or a geometric series.)

Suppose \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{i=1}^{\infty} b_n \) are series of positive terms.

a.) If \( \sum_{n=1}^{\infty} b_n \) is convergent and \( a_n \leq b_n \) for all \( n \), then \( \sum_{n=1}^{\infty} a_n \) is also convergent.

b.) If \( \sum_{n=1}^{\infty} b_n \) is divergent and \( a_n \geq b_n \) for all \( n \), then \( \sum_{n=1}^{\infty} a_n \) is also divergent.

6. **The Limit Comparison Test:** Conditions for using this test are the same conditions as the comparison test.

Suppose \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) are series of positive terms.

a.) If \( \lim_{n \to \infty} \frac{a_n}{b_n} = c > 0 \), then either both series converge or both diverge.

7. **The Alternating Series Test:** If the alternating series \( \sum_{n=1}^{\infty} (-1)^n a_n \) satisfies

a.) \( a_{n+1} \leq a_n \) for all \( n \) (ie the sequence \( \{a_n\} \) is decreasing).

b.) \( \lim_{n \to \infty} a_n = 0 \)

then the series converges.
8. **The Ratio Test**: (Use this test if the series contains \( n! \) or numbers raised to the \( n \)th power, such as \( 2^n \). If the ONLY number raised to the \( n \)th power is \((-1)^n\), then use the alternating series test).

   a.) If \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1 \), then the series \( \sum_{n=1}^{\infty} a_n \) is absolutely convergent (and therefore convergent).

   b.) If \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1 \) or \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty \), then the series \( \sum_{n=1}^{\infty} a_n \) is divergent.

   c.) If \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1 \), then the test fails.

9. **Remainder formulas:**

   - **The Remainder Estimate for the Integral test**: Suppose \( \sum_{n=1}^{\infty} a_n \) is a series which was shown to be convergent as a result of the integral test or a comparison test. This means that the sum of the series is finite. Let’s say \( \sum_{n=1}^{\infty} a_n = s \). Suppose further that I used a partial sum \( s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n \) to approximate \( s \). Then the remainder is defined to be \( R_n = \sum_{i=n+1}^{\infty} a_i = a_{n+1} + a_{n+2} + \ldots \).

     a.) If we want to get an upper bound for the error in using \( s_n \) to approximate \( s \), then

        \[
        R_n \leq \int_n^{\infty} f(x) \, dx.
        \]

     b.) If we want to get an interval on which the remainder lies, then

        \[
        \int_{n+1}^{\infty} f(x) \, dx \leq R_n \leq \int_n^{\infty} f(x) \, dx.
        \]

   - **The Alternating Series Theorem**: If \( \sum_{n=1}^{\infty} (-1)^n a_n \) is a convergent alternating series, and I used a partial sum \( s_n = \sum_{i=1}^{n} (-1)^i a_i \) to approximate the sum, then an upper bound on the absolute value of the remainder is \( |R_n| \leq a_{n+1} \).