Spring 2009 Math 152/STEPS
Series Practice
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Determine whether the series is absolutely convergent, conditionally convergent, or divergent. Use any tests you wish that are applicable.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+3^n}$
2. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$
3. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+2}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 2^{n-1}}{(-5)^n}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n+1}$
6. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$
7. $\displaystyle\sum_{k=1}^{\infty} \frac{2^k k!}{(k+2)!}$
8. $\displaystyle\sum_{k=1}^{\infty} k^2 e^{-k}$
9. $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n^3}$
10. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln n}$
11. $\displaystyle\sum_{n=1}^{\infty} \sin n$
12. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$
13. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin 2n}{1+2^n}$
14. $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{2\cdot 5\cdot 8\cdots(3n+2)}$
15. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+1}$
16. $\displaystyle\sum_{n=1}^{\infty} (-1)^n 2^{1/n}$
17. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$
18. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$
19. $\displaystyle\sum_{k=1}^{\infty} \frac{k+5}{5^k}$
20. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^2-1}}{n^3+2n^2+5}$
21. $\displaystyle\sum_{n=1}^{\infty} \tan(1/n)$
22. $\displaystyle\sum_{n=1}^{\infty} n\sin(1/n)$
23. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$
24. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{5^n}$
25. $\displaystyle\sum_{k=1}^{\infty} \frac{k \ln k}{(k+1)^3}$
26. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$
27. $\displaystyle\sum_{j=1}^{\infty} (-1)^j \frac{\sqrt{j}}{j+5}$
28. $\displaystyle\sum_{k=1}^{\infty} \frac{5^k}{3^k+4^k}$
29. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$
30. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+n\cos^2 n}$

① sequences: Finding the limit of a sequence.

② Finding the sum of a series

a.) given a formula for S_n, the n^{th} partial sum, in which case

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n$$

b.) geometric series:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad \text{if } |r|<1$$

c.) telescoping sum:

$$\sum_{n=1}^{\infty} (a_n - a_{n+1}), \quad \sum_{n=1}^{\infty} (a_n - a_{n+2}), \ldots$$

③ convergence Tests:

(a) positive series
- integral test
- comparison test
- limit comparison test
- ratio test [$n!$ or a^n]

(b) alternating series
- ratio test [$n!$ or a^n]
- alternating series test

(4) estimating sums: use a partial sum, s_n, to approximate the sum of a series

Remainder estimate:

(a) $R_n \le \int_n^{\infty} f(x)\,dx$, where $f(n) = a_n$

use this when the series is positive.

(b) $|R_n| \le a_{n+1}$, where a_{n+1} = positive part of $n+1^{th}$ term

use this when series is alternating

(5) power series: a series containing the variable x. use ratio test to find the interval & radius of convergence.

(6) representing a function as a power series: use $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $|x|<1$.

(7) Taylor series at $x = a$: $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

maclaurin: $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

8. Taylor polynomial at $x = a$ is:

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Taylor's inequality:

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

where $M = \max|f^{(n+1)}(x)|$ x in an interval containing a

① Finding the limit of a sequence:

$a_n = 1 + \arccos\left(\frac{n}{2n+1}\right)$ $\rightarrow \frac{1}{2}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1 + \arccos\left(\frac{n}{2n+1}\right)\right)$$

$$= 1 + \arccos\left(\frac{1}{2}\right) = \boxed{1 + \frac{\pi}{3}}$$

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② Finding the sum of a series

i. given $S_n = \frac{n+1}{4n+2}$ is a formula for the n^{th} partial sum of the series $\sum_{n=1}^{\infty} a_n$. Find $\sum_{n=1}^{\infty} a_n$.

$$\boxed{\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n}$$

(i) $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \frac{n+1}{4n+2}$

$= \boxed{\frac{1}{4}}$

(ii) $\sum_{n=1}^{\infty} \frac{2^{n+1}}{5^n} = \sum_{n=1}^{\infty} \frac{2 \cdot 2^n}{5^n}$ $\quad r = \frac{2}{5}$ ✓

$= \sum_{n=1}^{\infty} 2\left(\frac{2}{5}\right)^n$

need power $n-1$

$= \sum_{n=1}^{\infty} \frac{4}{5}\left(\frac{2}{5}\right)^{n-1} = \frac{4/5}{1 - 2/5}$

(iii) telescoping

$$\sum_{n=3}^{\infty}\left(\frac{1}{n+3}-\frac{1}{n+5}\right) \quad \underline{\underline{or}} \quad \sum_{n=2}^{\infty}\frac{1}{n(n+1)}$$

$\downarrow$ PFD

do $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(n+1)}$:

PFD: $$\frac{1}{n(n+1)} = \frac{\cancel{A}^{\,1}}{n} + \frac{\cancel{B}^{\,-1}}{n+1}$$

$$1 = A(n+1) + Bn$$

$n=0$: $1 = A$

$n=-1$: $1 = B(-1) \Rightarrow B=-1$

$$\sum_{n=2}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$S_n = \frac{1}{2} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{5}} + \dots \cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}} + \cancel{\frac{1}{n}} - \frac{1}{n+1}$$

$$S_n = \frac{1}{2} - \frac{1}{n+1}$$

$$\sum_{n=2}^{\infty}\frac{1}{n(n+1)} = \lim_{n\to\infty} S_n$$

$$= \lim_{n\to\infty}\left(\frac{1}{2}-\frac{1}{n+1}\right)$$

$$= \boxed{1/2}$$

Test For Divergence:

if $\lim_{n\to\infty} a_n \neq 0$, $\sum_{n=1}^{\infty} a_n$ diverges

if $\lim_{n\to\infty} a_n = 0$, then the test fails

#2. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+1}$

T.D. $\lim_{n\to\infty} \frac{(-1)^n n}{n+1}$ $\to 1$ if n even, $\to -1$ if n odd

$\frac{n}{n+1} \to 1$

$\lim_{n\to\infty} \frac{(-1)^n n}{n+1} \neq 0 \Rightarrow$ series diverges by T.D

#11. $\sum_{n=1}^{\infty} \sin n$:

T.D. $\lim_{n\to\infty} \sin(n) \neq 0$ diverges by T.D.

#22. $\sum_{n=1}^{\infty} n \sin \frac{1}{n}$

$\lim_{n\to\infty} n \sin\frac{1}{n} = \lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}}$ $\frac{0}{0}$

$\overset{L}{=} \lim_{n\to\infty} \frac{\cos\frac{1}{n}\left(-\cancel{\frac{1}{n^2}}\right)}{-\cancel{\frac{1}{n^2}}}$

$= \cos(0) = 1 \neq 0$

Diverges by T.D.

③ convergence tests

(a) positive series: if a positive series converges, then it automatically converges absolutely.

#1. $\sum_{n=1}^{\infty} \frac{1}{n+3^n} \leq \sum_{n=1}^{\infty} \frac{1^n}{3^n} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$

convergent geometric series

larger converges, so does smaller by C.T.

#5. $\sum_{n=1}^{\infty} \frac{1}{2n+1} < \sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$

divergent p-series

larger diverges C.T. Fails.

LCT: $\lim_{n\to\infty} \frac{a_n}{b_n}$ original, dominating terms

$= \lim_{n\to\infty} \frac{\frac{1}{2n+1}}{\frac{1}{2n}} = \lim_{n\to\infty} \frac{2n}{2n+1} = 1 > 0$ proceed...

conclusion. both series "do the same thing"

$\sum \frac{1}{2n}$ diverged p-series $\Rightarrow \sum \frac{1}{2n+1}$ also diverges by L.C.T.

#8. $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n} = \sum_{n=1}^{\infty} \frac{n^2}{e^n}$

contains exponential $\Rightarrow$ use ratio test!

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2}{e^{n+1}}\frac{e^n}{n^2}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)^2}{e\cdot \cancel{e^n}}\frac{\cancel{e^n}}{n^2}\right|$$

$$= \lim_{n\to\infty}\left|\frac{1}{e}\frac{(n+1)^2}{n^2}\right| \quad \to 1$$

$$= \frac{1}{e} < 1$$ series converges (absolutely)

#9. $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n^3}$ I.T.

$u = -x^3$ < $x=t,\ u=-t^3$; $x=1,\ u=-1$

$du = -3x^2\,dx$

$$\int_1^{\infty} x^2 e^{-x^3}\,dx$$

$$\lim_{t\to\infty}\int_1^{t} x^2 e^{-x^3}\,dx$$

$$\lim_{t\to\infty}\int_{-1}^{-t^3} -\frac{1}{3}e^{u}\,du$$

$$\lim_{t\to\infty} -\frac{1}{3}e^{u}\Big|_{-1}^{-t^3}$$

$e^{-\infty} = \frac{1}{e^{\infty}} = 0$

$$\lim_{t\to\infty} -\frac{1}{3}\left[e^{-t^3} - e^{-1}\right]$$

$$= \frac{1}{3}e^{-1}$$

integral conv. so does series by I.T.

#30. $\sum_{n=1}^{\infty} \frac{1}{n+n\cos^2 n}$ $\cos^2 n < 1$

$\frac{1}{\cos^2 n} > 1$

$$\sum_{n=1}^{\infty} \frac{1}{n(1+\underset{\downarrow}{\cos^2} n)} \geq \sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$$

$\quad\quad\quad\quad\quad\quad 1$

divergent p-series

smaller diverges so does larger by C.T.

(b) alternating series: if an alternating series converges, it <u>may</u> converge absolutely.

#3. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+2}$ AST: $a_n = \frac{n}{n^2+2}$

(i) $a_{n+1} \leq a_n$

(ii) $\lim_{n\to\infty} a_n = 0$

(i) show $\left(\frac{n}{n^2+2}\right)' < 0$

(ii) $\lim_{n\to\infty} \frac{n}{n^2+2} = 0$ higher power on bottom.

$\therefore \sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+2}$ converges

But does it converge absolutely?

to determine absolute convergence,

look at $\sum_{n=1}^{\infty} \left| (-1)^n \frac{n}{n^2+2} \right|$

CT $\sum_{n=1}^{\infty} \frac{n}{n^2+2} \leq \sum_{n=1}^{\infty} \frac{n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n}$

larger diverges
test fails.

LCT $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n}{n^2+2}}{\frac{1}{n}}$

$= \lim_{n\to\infty} \frac{n^2}{n^2+2} = 1 > 0$
proceed

Both series do the same
since $\sum \frac{1}{n}$ diverges so does $\sum \frac{n}{n^2+2}$ by L.C.T.

so $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+2}$ converges but not absolutely

#16. $\displaystyle\sum_{n=1}^{\infty} (-1)^n 2^{\frac{1}{n}}$

$2^{\frac{1}{\infty}} = 2^0 = 1$

$\displaystyle\lim_{n\to\infty} (-1)^n 2^{\frac{1}{n}}$ $\to 1$ if n even

$\to -1$ if n odd

$\displaystyle\lim_{n\to\infty} (-1)^n 2^{\frac{1}{n}} \neq 0$ series diverges by test for div.